Ma 227 **Final Exam Solutions** **11 May 1998**

Part I: Answer **all** questions.

Problem 1

a) (**8 points**)

Find the first four nonzero terms of the Fourier cosine series of

$$f(x) = \begin{cases} -1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

Solution

If $f(x)$ is a function defined on $[0, L]$, then its Fourier cosine expansion is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{1}{L}\int_0^L f(x)dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n = 1, 2, 3, \ldots$

Here $L = \pi$ so that $f(x) = \sum_{n=1}^{\infty} a_n \cos(nx), a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx.$

Thus $a_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(-1)dx + \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(0)dx = -\frac{1}{2}.$ Also,

$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}(-1)\cos nx dx = -\frac{2}{n\pi}[\sin nx]_0^{\frac{\pi}{2}} = -\frac{2}{n\pi}\left[\sin\frac{n\pi}{2}\right]$$

Therefore

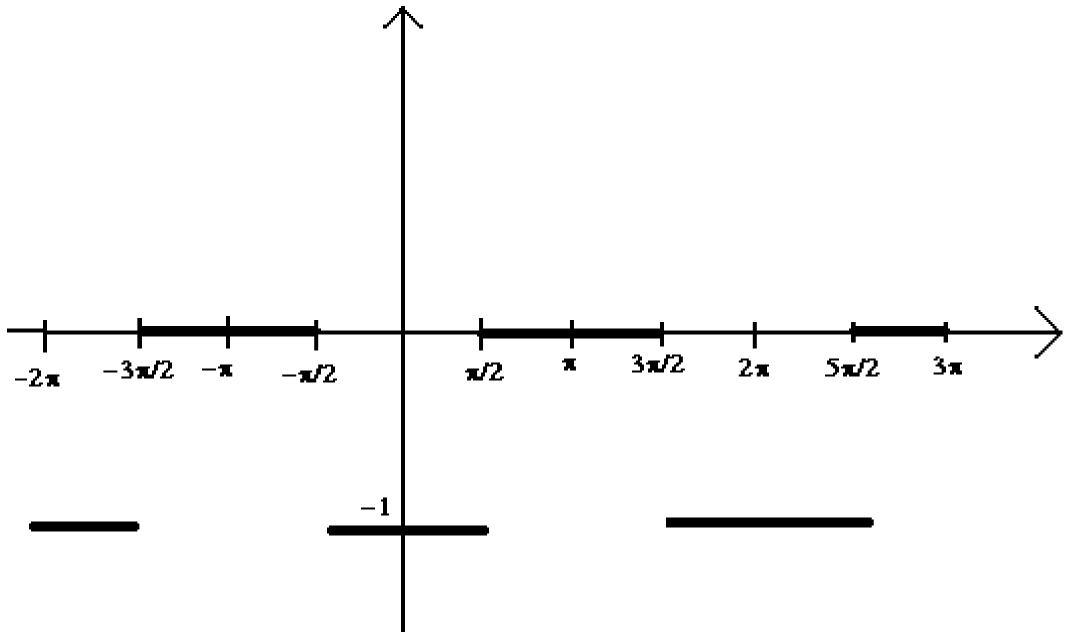
$a_1 = -\frac{2}{\pi}, \quad a_2 = 0, \quad a_3 = +\frac{2}{3\pi}, \quad a_4 = 0, \quad a_5 = -\frac{2}{5\pi}, \quad a_6 = 0, \quad a_7 = +\frac{2}{7\pi}$

Hence

$$f(x) = -\frac{1}{2} - -\frac{2}{\pi}\cos x + 0 \bullet \cos 2x + \frac{2}{3\pi}\cos 3x + 0 \bullet \cos 4x - \frac{2}{5\pi}\cos 5x + 0 \bullet \cos 6x + \frac{2}{7\pi}\cos 7x$$

b) (**8 points**)

Sketch the graph of the function to which the Fourier series in (a) converges on $-2\pi < x < 3\pi$.



c) (**9 points**)
Find the eigenvalues and eigenfunctions for the problem

$y'' + \lambda y = 0$; $y(0) = 0$; $y(2) = 0$

Be sure to check the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.
I. Consider the case $\lambda < 0$ first. Let $\lambda = -\alpha^2$ where $\alpha \neq 0$. The DE becomes

$$y'' - \alpha^2 y = 0.$$

The general solution of this equation is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. Thus

$$y(0) = c_1 + c_2 = 0 \text{ and } y(2) = c_1 e^{2\alpha} + c_2 e^{-2\alpha} = 0.$$

The first equation implies that $c_1 = -c_2$. Thus the second equation becomes $c_1(e^{2\alpha} + e^{-2\alpha}) = 0$. Thus $c_1 = 0$; this tells us that $c_2 = 0$ also. Therefore $y = 0$ is the only solution if $\lambda < 0$. Hence there are no negative eigenvalues.

II. Suppose $\lambda = 0$. The DE becomes $y'' = 0$ which has the solution $y = c_1 x + c_2$. The boundary conditions imply $y(0) = c_1 = 0$, so that $y = c_2$. But $y(2) = c_2 = 0$ so that $y = 0$. Hence there is no eigenfunction corresponding to the eigenvalue $\lambda = 0$.

III. Suppose $\lambda > 0$. Let $\lambda = \beta^2$ where $\beta \neq 0$. The DE becomes

$$y'' + \beta^2 y = 0.$$

The general solution of this equation is $y(x) = c_1 \sin \beta x + c_2 \cos \beta x.$ Thus

Now $y(0) = c_2 = 0$ Thus $y(x) = c_2 \sin \beta x.$ Now $y(2) = c_2 \sin 2\beta = 0.$ For a nontrivial solution we must have $c_2 \neq 0.$ This means that $\sin 2\beta = 0$ or $\beta = \frac{n\pi}{2},\ n = 1, 2, 3, \ldots$ The eigenvalues are therefore $\lambda = \beta^2 = \frac{n^2\pi^2}{4}$ and the corresponding eigenfunctions are $y_n = a_n \sin \frac{n\pi}{2} x,\ n = 1, 2, 3, \ldots$

Problem 2

a) (**10 points**)

Use separation of variables, $u(x,t) = X(x)T(t)$, to find ordinary differential equations which $X(x)$ and $T(t)$ must satisfy if $u(x,t)$ is to be a solution of

$$5x^5 t^2 u_{tt} + (t+3)^5 (x+5)^2 u_{xx} = 0$$

Do not solve these equations.

Solution:

$u_x = X'T, \qquad u_{xx} = X''T, \qquad u_t = XT', \quad u_{tt} = XT''$

Thus the given equation becomes

$$15t^2 x^5 XT'' + (t+3)^5 (x+5)^2 X''T = 0$$

$$\Rightarrow \qquad 15x^5 \frac{X}{(x+5)^2 X''} = -(t+3)^5 \frac{T}{t^2 T''} = k, \qquad k \text{ a constant}$$

This yields the two ODEs

$$15x^5 X - k(x+5)^2 X'' = 0$$

$$(t+3)^5 T + kt^2 T'' = 0$$

b) (**15 points**)

Solve:

P.D.E.: $u_{xx} = 4u_t$ B.C.'s: $u(0,t) = u(2,t) = 0$

I.C.: $u(x,0) = -3\sin \frac{\pi x}{2} + 23 \sin \pi x - 4 \sin 2\pi x$

Let $u(x,t) = X(x)T(t).$ Then differentiating and substituting in the PDE yields

$$X''T = 4XT'$$

$$\Rightarrow \qquad \frac{X''}{X} = 4\frac{T'}{T}$$

Using the argument that the left hand side is purely a function of x and the right hand side is purely a function of t, and the only way that they can be equal is if they are equal to a constant, we get

$$\frac{X''}{X} = 4\frac{T'}{T} = k \qquad k \text{ a constant}$$

This yields the two *ordinary differential equations*

$$X'' - kX = 0 \qquad \text{and} \qquad T' - \frac{1}{4}kT = 0$$

The boundary condition $u(0,t) = 0$ implies that $X(0)T(t) = 0.$ We cannot have $T(t) = 0$, since this would imply that $u(x,t) = 0$. Thus $X(0) = 0.$ Similarly, the boundary condition $u(2,t) = 0$ leads to $X(2) = 0.$

We now have the following boundary value problem for $X(x)$:

$$X'' - kX = 0 \qquad X(0) = X(2) = 0$$

This boundary value problem is the one given in Problem 1(c) above with $k = -\lambda$.The solution is

$$k = -(\tfrac{n\pi}{2})^2 \qquad X_n(x) = a_n \sin\tfrac{n\pi}{2}x \qquad n = 1,2,3,\ldots$$

Substituting the values of k into the equation for $T(t)$ leads to

$$T' + \frac{n^2\pi^2}{16}T = 0$$

which has the solution $T_n(t) = c_n e^{-\frac{n^2\pi^2 t}{16}},\ n = 1,2,3,\ldots$

We now have the solutions

$$u_n(x,t) = A_n \sin\tfrac{n\pi}{2}x e^{-\frac{n^2\pi^2 t}{16}} \qquad n = 1,2,3,\ldots$$

Since the boundary conditions and the equation are linear and homogeneous, it follows that

$$u(x,t) = \textstyle\sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\tfrac{n\pi}{2}x e^{-\frac{n^2\pi^2 t}{16}}$$

satisfies the PDE and the boundary conditions. Since

$$u(x,0) = -3\sin\frac{\pi x}{2} + 23\sin\pi x - 4\sin 2\pi x = \textstyle\sum_{n=1}^{\infty} A_n \sin\tfrac{n\pi}{2}x.$$

Matching the cosine terms on both sides of this equation leads to

$A_1 = -3 \qquad A_2 = 23$ and $A_4 = -4$. All of the other constants must be zero, since there are no sine terms on the left to match with them. Thus

$$u(x,t) = -3\sin\frac{\pi x}{2}e^{-\frac{\pi^2}{16}t} + 23\sin\pi x e^{-\frac{\pi^2}{4}t} - 4\sin 2\pi x e^{-\pi^2 t}$$

Problem 3

a) (**15 points**)

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$.

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2-\lambda & 0 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-1)^{1+1}\begin{vmatrix} -\lambda & 1 \\ -4 & 5-\lambda \end{vmatrix} + (2-\lambda)(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 4 & 5-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-5\lambda + \lambda^2 + 4) - (2-\lambda)(5-\lambda-4) = (2-\lambda)[\lambda^2 - 4\lambda + 3] = (2-\lambda)(\lambda-3)(\lambda-1)$$

Hence the eigenvalues are $\lambda = 1, 2, 3$. The system of equations $(A - \lambda I)X = 0$ for this problem is

$$\begin{aligned} (1-\lambda)x_1 + 2x_2 - x_3 &= 0 \\ x_1 - \lambda x_2 + x_3 &= 0 \\ 4x_1 - 4x_2 + (5-\lambda)x_3 &= 0 \end{aligned}$$

$\lambda = 1 \Rightarrow$

$$\begin{aligned} 2x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 4x_1 - 4x_2 + 4x_3 &= 0 \end{aligned}$$

This system has the solution $x_3 = 2x_2,\ \ x_1 = x_2 - x_3 = -x_2$. The eigenvector is therefore $\begin{matrix} 1 \\ -1 \\ 2 \end{matrix}$. Similarly we have for $\lambda = 2$ $\begin{matrix} 1 \\ -\frac{1}{2} \\ -2 \end{matrix}$ and for $\lambda = 3$ $\begin{matrix} 1 \\ -1 \\ -4 \end{matrix}$.

b) (**10 points**)

Find the solution, if it exists, of

$$\begin{aligned} x_1 + 2x_2 - 2x_3 + 3x_4 - 4x_5 &= -3 \\ 2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 &= -1 \\ -x_1 - 2x_2 \qquad - 3x_4 + 11x_5 &= 15 \end{aligned}$$

Solution:

$$\begin{matrix} 1 & 2 & -2 & 3 & -4 & -3 \\ 2 & 4 & -5 & 6 & -5 & -1 \\ -1 & -2 & 0 & -3 & 11 & 15 \end{matrix} \xrightarrow[-2R_1+R_2]{R_1+R_3} \begin{matrix} 1 & 2 & -2 & 3 & -4 & -3 \\ 0 & 0 & -1 & 0 & 3 & 5 \\ 0 & 0 & -2 & 0 & 7 & 12 \end{matrix} \xrightarrow[(-1)R_2]{-2R_2+R_3} \begin{matrix} 1 & 2 & -2 & 3 & -4 & -3 \\ 0 & 0 & 1 & 0 & -3 & -5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{matrix}$$

Since the rank of the coefficient matrix equals the rank of the augmented matrix, there exists a solution. It is

$x_5 = 2 \qquad x_3 - 3x_5 = -5$ or $x_3 = -5 + 3x_3 = -5 + 6 = 1$ and $x_1 = -2x_2 - 3x_4 + 7$

Problem 4

a) (**13 points**)

Verify Green's theorem when $P = 4x - 2y$; $Q = 2x + 6y$ and C is the ellipse $x = 2\cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$. (Recall that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab.$)

SOLUTION

For this ellipse, $a = 2$ and $b = 1$. Let G be the interior of C. Green's theorem states that the two integrals $\oint_C Pdx + Qdy$ and $\iint_G (Q_x - P_y)dxdy$ are equal. We must verify this.

Since $Q_x = 2$ and $P_y = -2$,

$$\iint_G (Q_x - P_y)dxdy = \iint_G 4dxdy = 4\iint_G dxdy = 4(Area\ of\ G) = 4(\pi)(2)(1) = 8\pi$$

The ellipse is already parametrized by θ. Since $dx = -2\sin\theta\, d\theta$ and $dy = \cos\theta\, d\theta$,

$$\oint_C Pdx + Qdy = \oint_C (4x - 2y)dx + (2x + 6y)dy$$

$$= \int_0^{2\pi} \{(8\cos\theta - 2\sin\theta)(-2\sin\theta) + (4\cos\theta + 6\sin\theta)(\cos\theta)\}d\theta$$

$$= \int_0^{2\pi} \{-16\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 6\sin\theta\cos\theta\}d\theta$$

$$= \int_0^{2\pi} \{4 - 10\sin\theta\cos\theta\}d\theta = 8\pi$$

The theorem has now been verified.

Problem 4

b) (**12 points**)

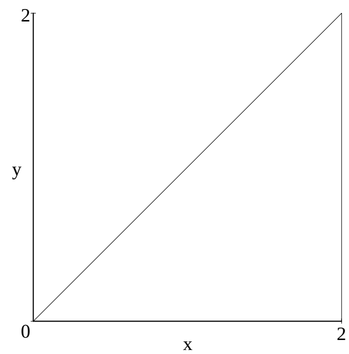
Consider $\int_0^2 \int_y^2 f(x,y)\,dx\,dy$.

a) Sketch the region of integration.

b) Write the integral reversing the order of integration.

c) Rewrite the integral in terms of polar coordinates.

SOLUTION



b) Taking the limits from the sketch, we get $\int_0^2 \int_0^x f(x,y)\,dy\,dx$

c) The limits on θ are clear from the sketch. Noting that the polar equation of the line $x = 2$ is $r\cos\theta = 2$ or $r = 2\sec\theta$, we have $\int_0^{\pi/4} \int_0^{2\sec\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$. Don't forget the extra factor of r inside the integral.

Problem 5

Consider the $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{F} = (2xyz + z^2y)\,\hat{i} + (x^2z + z^2x)\,\hat{j} + (x^2y + 2xyz)\,\hat{k}$

a) (**12 points**)

Show that $\nabla \times \vec{F} = \vec{0}$. What does this tell you about $\oint_C \vec{F} \cdot \vec{dr}$, where C is any closed curve?

SOLUTION

$$\nabla \times \vec{F} = curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz + z^2y & x^2z + z^2x & x^2y + 2xyz \end{vmatrix}$$

$$= (x^2 + 2xz - x^2 - 2zx)\,\hat{i} - (2xy + 2yz - 2xy - 2zy)\,\hat{j} + (2xz + z^2 - 2xz - z^2)\,\hat{k} = \vec{0}$$

Then $\oint_C \vec{F} \cdot \vec{dr} = 0$ for any closed curve C.

Or, equivalently, $\oint_C \vec{F} \cdot \vec{dr}$ is independent of the path taken between two given points.

b) (**13 points**)

Find a function $\Phi(x,y,z)$ such that $\nabla\Phi = \vec{F}$

SOLUTION

$\vec{F} = \nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}$ We set equal the corresponding components.

$$\frac{\partial\Phi}{\partial x} = 2xyz + z^2y \quad \Rightarrow \quad \Phi(x,y,z) = x^2yz + xyz^2 + C_1(y,z)$$
$$\frac{\partial\Phi}{\partial y} = x^2z + z^2x \quad \Rightarrow \quad \Phi(x,y,z) = x^2yz + xyz^2 + C_2(x,z)$$
$$\frac{\partial\Phi}{\partial z} = x^2y + 2xyz \quad \Rightarrow \quad \Phi(x,y,z) = x^2yz + xyz^2 + C_3(x,y)$$

Comparing the three expression for Φ, we let $C_1 = C_2 = C_3 = C$. Since C_1 is independent of x, so is C. Likewise, C_2 is independent of y, and C_3 of z. Therefore C is independent of all variables, i.e. it is a constant.

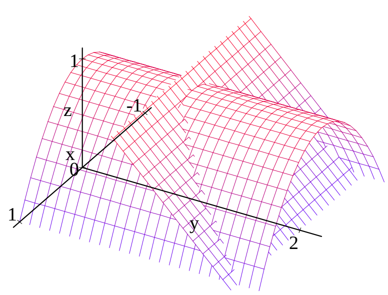
Finally, we have $\Phi(x,y,z) = x^2yz + xyz^2 + C$

Problem 6

a) (**12 points**)

Let S be the closed surface bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, $y = z = 2$. Sketch S.

SOLUTION



b) (**13 points**)

Let S be the closed surface in 6 a), $\vec{n}$ the outward unit normal to S, and $\vec{F} = xy\hat{i} + (y^2 + e^{xz^2})\hat{j} + \sin xy\hat{k}$. Use the Divergence Theorem to transform the $\iint_S \vec{F} \cdot \vec{n}\, dS$ into a triple integral. Do not evaluate the integral.

SOLUTION

$$\iint_S \vec{F}\cdot\vec{n}dS = \iiint_V div\vec{F}\ dV = \iiint_V (y+2y+0)\ dV = \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} 3y\ dydzdx$$

The limits can be deduced from the sketch. Other correct expressions include:

$$\int_0^1\int_{-\sqrt{1-z}}^{+\sqrt{1-z}}\int_0^{2-z} 3y\ dydxdz \qquad \int_0^1\int_0^{2-z}\int_{-\sqrt{1-z}}^{+\sqrt{1-z}} 3y\ dxdydz$$

There is no way to express the integral in the forms $dzdydx$ or $dzdxdy$ or $dxdzdy$ without splitting the integral into two smaller integrals.

Problem 7

a) (**10 points**)

Without expanding show that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

SOLUTION:

Using elementary operations on the columns we have,

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} \to^{-C_2+C_1} \begin{vmatrix} 1 & 0 & b+c \\ 1 & b-a & c+a \\ 1 & c-a & a+b \end{vmatrix} \to^{C_2+C_3} \begin{vmatrix} 1 & 0 & b+c \\ 1 & b-a & b+c \\ 1 & c-a & b+c \end{vmatrix}$$

$$= (b+c)\begin{vmatrix} 1 & 0 & 1 \\ 1 & b-a & 1 \\ 1 & c-a & 1 \end{vmatrix} = 0$$

Since the first and third columns are the same.

b) (**15 points**)

Use Stokes' Theorem to compute the integral $\iint_S curl\vec{F}\cdot\vec{n}dS$, where $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$, and S is the part of sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2 = 1$ and above the xy –plane. Sketch S.

(Note: $\cos^2 t - \sin^2 t = \cos 2t$.)

SOLUTION:

Stoke's Theorem states that $\iint_S curl\vec{F}\cdot\vec{n}dS = \int_{\partial S}\vec{F}\cdot d\vec{r}$. We want to find $\int_{\partial S}\vec{F}\cdot d\vec{r}$.

The region S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and is above the xy –plane. To find where the sphere and the cylinder intersect we set $x^2+y^2=1$ in the equation $x^2+y^2+z^2=4$. This yields $1+z^2=4$ or $z=\sqrt{3}$. Thus ∂S is given by $x^2+y^2=1,\ z=\sqrt{3}$. We parametrize this as

$$x=\cos t,\ y=\sin t,\ z=\sqrt{3} \qquad 0\le t\le 2\pi$$

Hence $\vec{r}(t)=\cos t\,\vec{i}+\sin t\,\vec{j}+\sqrt{3}\ \vec{k}$ and $\vec{r}'(t)=-\sin t\,\vec{i}+\cos t\,\vec{j}+0\vec{k}$ and $\vec{F}(t)=\sqrt{3}\ \sin t\,\vec{i}+\sqrt{3}\ \cos t\,\vec{j}+\cos t\sin t\,\vec{k}$

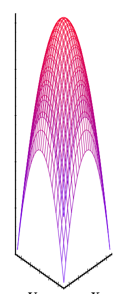
$$\int_{\partial S}\vec{F}\cdot d\vec{r}.=\int_0^{2\pi}\vec{F}(t)\cdot\vec{r}'(t)dt=\int_0^{2\pi}\sqrt{3}\,(\cos t-\sin t)\ dt=\sqrt{3}\int_0^{2\pi}\cos 2t\ dt=0$$

Problem 8

a) (**10 points**)

Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$. Sketch the volume.

SOLUTION:



The paraboloid $z=1-x^2-y^2$ intersects the x,y –plane on the circle $x^2+y^2=1$. Let D denote the inside of the circle. Then the volume is

$$V=\iint_D\int_0^{1-x^2-y^2}dzdA$$

Using cylindrical coordinates $x=r\cos\theta,\ y=r\sin\theta,\ z=z$ we have,

$$V=\int_0^{2\pi}\int_0^1\int_0^{1-r^2}r\ dzdrd\theta=\int_0^{2\pi}\int_0^1(1-r^2)rdrd\theta=\tfrac{\pi}{2}$$

b) (**15 points**)

Find the eigenvalues and eigenfunctions of

$$y''+\lambda y=0 \qquad y(-\pi)=y(\pi) \qquad y'(-\pi)=y'(\pi)$$

SOLUTION:

We must consider the cases $\lambda < 0, \lambda = 0, \lambda > 0.$

Case I. $\lambda < 0$ Let $\lambda = -\alpha^2$

$$y'' - \alpha^2 y = 0$$

The auxiliary equation is $r^2 - \alpha^2 = 0$ which tells us that $r = \pm\alpha$ real distinct roots.

The characteristic equation is

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$y' = \alpha c_1 e^{\alpha x} - \alpha c_2 e^{-\alpha x}$$

$y(-\pi) = y(\pi) \quad \Rightarrow c_1 e^{-\alpha\pi} + c_2 e^{\alpha\pi} = c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} \Rightarrow c_1(e^{\alpha\pi} - e^{-\alpha\pi}) = c_2(e^{\alpha\pi} - e^{-\alpha\pi})$

$$\Rightarrow c_1 = c_2$$

$y'(-\pi) = y'(\pi) \qquad \Rightarrow \alpha c_1 e^{-\alpha\pi} - \alpha c_2 e^{\alpha\pi} = \alpha c_1 e^{\alpha\pi} - \alpha c_2 e^{-\alpha\pi}$

But $c_1 = c_2$ so $2c_1 e^{\alpha\pi} = 2c_1 e^{-\alpha\pi}. \Rightarrow c_1 = 0$ so that $c_2 = 0.$ Thus $y = 0$ is the only solution, and there are no negative eigenvalues.

Case II. $\lambda = 0$

$$y'' = 0$$

The auxiliary equation is $r^2 = 0$, a repeated root.

$$y = c_1 x + c_2$$

$y(-\pi) = y(\pi) \Rightarrow -c_1\pi + c_2 = c_1\pi + c_2 = 0$

This implies that $c_1 = 0.$ This also satisfies $y'(-\pi) = y'(\pi)$ so $y = c_2$ where $c_2 \neq 0$ is a nontrivial solution corresponding to $\lambda = 0.$

Case III. $\lambda > 0$ Let $\lambda = \alpha^2,$ where $\alpha \neq 0.$ The DE is then

$$y'' + \alpha^2 y = 0$$

The auxiliary equation is $r^2 + \alpha^2 = 0$ which tells us that $r = \pm\alpha i$ complex roots.

The characteristic equation is

$$y = c_1 \cos\alpha x + c_2 \sin\alpha x$$

$$y' = -c_1\alpha \sin\alpha x + c_2\alpha \cos\alpha x$$

$y(-\pi) = y(\pi) = 0 \Rightarrow c_1\cos(-\alpha\pi) + c_2\sin(-\alpha\pi) = c_1\cos\alpha\pi + c_2\sin\alpha\pi$ or

$c_1\cos(\alpha\pi) - c_2\sin(\alpha\pi) = c_1\cos\alpha\pi + c_2\alpha\sin\alpha\pi \Rightarrow 2c_2\sin\alpha\pi = 0$

$y'(-\pi) = y'(\pi)$ implies that $-c_1\alpha\sin(-\alpha\pi) + c_2\alpha\cos(-\alpha\pi) = -c_1\sin\alpha\pi + c_2\cos\alpha\pi$
$\Rightarrow 2c_1\alpha\sin\alpha\pi = 0.$ Thus if $\sin\alpha\pi \neq 0,$ since we have $c_1 = c_2 = 0.$ Hence for a nonzero solution we must have $\sin\alpha\pi = 0.$

This is true when $\alpha = n,$ where $n = \pm1, \pm2, \pm3, \ldots..$

Therefore the eigenvalues are $\lambda_n = n^2$ and the eigenfunctions are
$y_n = a_n\cos n\pi x + b_n\sin n\pi x.$

Problem 9

a) (**13 points**)**Suppose that** $f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x)$, **where** $\{\phi_1, \phi_2, \phi_3, \ldots\}$ **is an orthonormal set on the interval** $[a, b]$. **Show that** $\int_a^b f^2(x)dx = \sum_{i=1}^{\infty} a_i^2$

SOLUTION:

By definition of orthonormality we know that

$$\int_a^b \varphi_i(x)\varphi_j(x)\ dx = <\varphi_i, \varphi_j> = \begin{cases} 1 \text{ if } i = j \\ \\ 0 \text{ if } i \neq j \end{cases}$$

$$\int_a^b f^2(x)dx = <f, f> = <\sum_{i=1}^{\infty} a_i\varphi_i(x), \sum_{j=1}^{\infty} a_j\varphi_j(x)>$$

$$= a_1^2 < \varphi_1(x), \varphi_1(x) > + a_1a_2 < \varphi_1(x), \varphi_2(x) > \\ + a_1a_3 < \varphi_1(x), \varphi_3(x) > + \cdots + a_2a_1 < \varphi_2(x), \varphi_1(x) > \\ + a_2^2 < \varphi_2(x), \varphi_2(x) > + a_2a_3 < \varphi_2(x), \varphi_3(x) > + \cdots$$

However, since we have an orthonormal set we have that

$$\int_a^b f^2(x)dx = a_1^2 < \varphi_1(x), \varphi_1(x) > + a_2^2 < \varphi_2(x), \varphi_2(x) > + \cdots = \sum_{i=1}^{\infty} a_i^2$$

b.) (**12 points**)

Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix}$., inverse: $\begin{bmatrix} -3 & 9 & 2 \\ 2 & -6 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

SOLUTION:

Using row reduction,

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \to^{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \to^{-2R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\to^{-6R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -6 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \to^{-2R_2 + R_1} \begin{bmatrix} 1 & 0 & 3 & -3 & 6 & -2 \\ 0 & 1 & 0 & 2 & -6 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{-3R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & -3 & -9 & -2 \\ 0 & 1 & 0 & 2 & -6 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The inverse is $\begin{bmatrix} -3 & 9 & 2 \\ 2 & -6 & -1 \\ 0 & 1 & 0 \end{bmatrix}$