Ma 227 Final Exam Solutions 11 May 1998

Part I: Answer all questions.

Problem 1

a) (8 points)
Find the first four nonzero terms of the Fourier cosine series of

\[ f(x) = \begin{cases} 
-1 & 0 < x < \frac{\pi}{2} \\
0 & \frac{\pi}{2} < x < \pi 
\end{cases} \]

Solution

If \( f(x) \) is a function defined on \([0, L]\), then its Fourier cosine expansion is given by

\[ f(x) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) \]

where \( a_0 = \frac{1}{L} \int_0^L f(x) \, dx \) and \( a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx \) \( n = 1, 2, 3, \ldots \)

Here \( L = \pi \) so that \( f(x) = \sum_{n=1}^{\infty} a_n \cos(nx) \), \( a_0 = \frac{1}{\pi} \int_0^\pi f(x) \, dx \) and \( a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \).

Thus \( a_0 = \frac{1}{\pi} \int_0^{\pi/2} (-1) \, dx + \frac{1}{\pi} \int_0^{\pi/2} (0) \, dx = -\frac{1}{2} \). Also,

\[ a_n = \frac{2}{\pi} \int_0^{\pi/2} (-1) \cos nx \, dx = -\frac{2}{n\pi} [\sin nx]_0^{\pi/2} = -\frac{2}{n\pi} \left[ \sin \frac{n\pi}{2} \right] \]

Therefore

\[ a_1 = -\frac{2}{\pi}, \quad a_2 = 0, \quad a_3 = +\frac{2}{3\pi}, \quad a_4 = 0, \quad a_5 = -\frac{2}{5\pi}, \quad a_6 = 0, \quad a_7 = +\frac{2}{7\pi} \]

Hence

\[ f(x) = -\frac{1}{2} - \frac{2}{\pi} \cos x + 0 \cdot \cos 2x + \frac{2}{3\pi} \cos 3x + 0 \cdot \cos 4x - \frac{2}{5\pi} \cos 5x + 0 \cdot \cos 6x + \frac{2}{7\pi} \cos 7x \]

b) (8 points)

Sketch the graph of the function to which the Fourier series in (a) converges on \(-2\pi < x < 3\pi\).
c) (9 points)
Find the eigenvalues and eigenfunctions for the problem

\[ y'' + \lambda y = 0; \quad y(0) = 0; \quad y(2) = 0 \]

Be sure to check the cases \( \lambda < 0, \lambda = 0, \) and \( \lambda > 0. \)

I. Consider the case \( \lambda < 0 \) first. Let \( \lambda = -\alpha^2 \) where \( \alpha \neq 0. \) The DE becomes

\[ y'' - \alpha^2 y = 0. \]

The general solution of this equation is \( y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}. \) Thus

\[ y(0) = c_1 + c_2 = 0 \quad \text{and} \quad y(2) = c_1 e^{2\alpha} + c_2 e^{-2\alpha} = 0. \]

The first equation implies that \( c_1 = -c_2. \) Thus the second equation becomes

\[ c_1(e^{2\alpha} + e^{-2\alpha}) = 0. \]

Thus \( c_1 = 0; \) this tells us that \( c_2 = 0 \) also. Therefore \( y = 0 \) is the only solution if \( \lambda < 0. \) Hence there are no negative eigenvalues.

II. Suppose \( \lambda = 0. \) The DE becomes \( y'' = 0 \) which has the solution \( y = c_1 x + c_2. \) The boundary conditions imply \( y(0) = c_1 = 0, \) so that \( y = c_2. \) But \( y(2) = c_2 = 0 \) so that \( y = 0. \) Hence there is no eigenfunction corresponding to the eigenvalue \( \lambda = 0. \)

III. Suppose \( \lambda > 0. \) Let \( \lambda = \beta^2 \) where \( \beta \neq 0. \) The DE becomes

\[ y'' + \beta^2 y = 0. \]
The general solution of this equation is \( y(x) = c_1 \sin \beta x + c_2 \cos \beta x \). Thus

Now \( y(0) = c_2 = 0 \) Thus \( y(x) = c_2 \sin \beta x \). Now \( y(2) = c_2 \sin 2\beta = 0 \). For a nontrivial solution we must have \( c_2 \neq 0 \). This means that \( \sin 2\beta = 0 \) or \( \beta = \frac{n\pi}{2} \), \( n = 1, 2, 3, \ldots \) The eigenvalues are therefore \( \lambda = \beta^2 = \frac{n^2\pi^2}{4} \) and the corresponding eigenfunctions are \( y_n = a_n \sin \frac{n\pi}{2} x, \ n = 1, 2, 3, \ldots \)

**Problem 2**

a) (10 points)

Use separation of variables, \( u(x, t) = X(x)T(t) \), to find ordinary differential equations which \( X(x) \) and \( T(t) \) must satisfy if \( u(x, t) \) is to be a solution of

\[
5x^5 t^2 u_{tt} + (t + 3)^5 (x + 5)^2 u_{xx} = 0
\]

Do not solve these equations.

**Solution:**

\[
\begin{align*}
  u_x &= X'T, & u_{xx} &= X''T, & u_t &= XT', & u_{tt} &= XT''
\end{align*}
\]

Thus the given equation becomes

\[
15t^2 x^5 XT'' + (t + 3)^5 (x + 5)^2 X''T = 0
\]

\[
\Rightarrow \quad 15x^5 \frac{X}{(x + 5)^2 X''} = -(t + 3)^5 \frac{T}{t^2 T''} = k, \quad k \text{ a constant}
\]

This yields the two ODEs

\[
15x^5 X - k(x + 5)^2 X'' = 0
\]

\[
(t + 3)^5 T + kt^2 T'' = 0
\]

b) (15 points)

Solve:

P.D.E.: \( u_{xx} = 4u_t \) \quad B.C.'s: \( u(0, t) = u(2, t) = 0 \)

I.C.: \( u(x, 0) = -3 \sin \frac{\pi x}{2} + 23 \sin \pi x - 4 \sin 2\pi x \)

Let \( u(x, t) = X(x)T(t) \). Then differentiating and substituting in the PDE yields

\[
\frac{X''T}{X} = 4 \frac{T'}{T}
\]

Using the argument that the left hand side is purely a function of \( x \) and the right hand side is purely a function of \( t \), and the only way that they can be equal is if they are equal to a constant, we get
\[
\frac{X''}{X} = 4 \frac{T'}{T} = k \quad \text{a constant}
\]

This yields the two ordinary differential equations

\[
X'' - kX = 0 \quad \text{and} \quad T' - \frac{1}{4} kT = 0
\]

The boundary condition \(u(0, t) = 0\) implies that \(X(0)T(t) = 0\). We cannot have \(T(t) = 0\), since this would imply that \(u(x, t) = 0\). Thus \(X(0) = 0\). Similarly, the boundary condition \(u(2, t) = 0\) leads to \(X(2) = 0\).

We now have the following boundary value problem for \(X(x)\):

\[
X'' - kX = 0 \quad X(0) = X(2) = 0
\]

This boundary value problem is the one given in Problem 1(c) above with \(k = -\lambda\). The solution is

\[
k = -\left(\frac{n\pi}{2}\right)^2 \quad X_n(x) = a_n \sin \frac{n\pi}{2} x \quad n = 1, 2, 3, \ldots
\]

Substituting the values of \(k\) into the equation for \(T(t)\) leads to

\[
T' + \frac{n^2 \pi^2}{16} T = 0
\]

which has the solution \(T_n(t) = c_n e^{-\frac{n^2 \pi^2}{16} t}, \quad n = 1, 2, 3, \ldots\)

We now have the solutions

\[
u_n(x, t) = A_n \sin \frac{n\pi}{2} x e^{-\frac{n^2 \pi^2}{16} t} \quad n = 1, 2, 3, \ldots
\]

Since the boundary conditions and the equation are linear and homogeneous, it follows that

\[
u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{2} x e^{-\frac{n^2 \pi^2}{16} t}
\]

satisfies the PDE and the boundary conditions. Since

\[
u(x, 0) = -3 \sin \frac{\pi x}{2} + 23 \sin \pi x - 4 \sin 2\pi x = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{2} x.
\]

Matching the cosine terms on both sides of this equation leads to
A_1 = -3 \quad A_2 = 23 \quad A_4 = -4. All of the other constants must be zero, since there are no sine terms on the left to match with them. Thus

\[ u(x, t) = -3 \sin \frac{\pi x}{2} e^{-\frac{\pi^2}{4} t} + 23 \sin \pi x e^{-\frac{\pi^2}{16} t} - 4 \sin 2\pi x e^{-\pi^2 t} \]

**Problem 3**

a) (15 points)

Find the eigenvalues and eigenvectors of

\[
A = \begin{bmatrix}
1 & 2 & -1 \\
1 & 0 & 1 \\
4 & -4 & 5
\end{bmatrix}
\]

\[
\begin{vmatrix}
1 - \lambda & 2 & -1 \\
1 & -\lambda & 1 \\
4 & -4 & 5 - \lambda
\end{vmatrix}
= (2 - \lambda)(-1)^{1+1}
\begin{vmatrix}
-\lambda & 1 \\
-4 & 5 - \lambda
\end{vmatrix}
+ (2 - \lambda)(-1)^{1+2}
\begin{vmatrix}
1 & 1 \\
4 & 5 - \lambda
\end{vmatrix}
\]

\[
= (2 - \lambda)(-5\lambda + \lambda^2 + 4) - (2 - \lambda)(5 - \lambda - 4) = (2 - \lambda)[\lambda^2 - 4\lambda + 3] = (2 - \lambda)(\lambda - 3)(\lambda - 1)
\]

Hence the eigenvalues are \( \lambda = 1, 2, 3 \). The system of equations \((A - \lambda I)X = 0\) for this problem is

\[
\begin{align*}
(1 - \lambda)x_1 + 2x_2 - x_3 &= 0 \\
x_1 - \lambda x_2 + x_3 &= 0 \\
4x_1 - 4x_2 + (5 - \lambda)x_3 &= 0
\end{align*}
\]

\( \lambda = 1 \Rightarrow \)

\[
\begin{align*}
2x_2 - x_3 &= 0 \\
x_1 - x_2 + x_3 &= 0 \\
4x_1 - 4x_2 + 4x_3 &= 0
\end{align*}
\]

This system has the solution \( x_3 = 2x_2, \; x_1 = x_2 - x_3 = -x_2 \). The eigenvector is

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

therefore \( -1 \). Similarly we have for \( \lambda = 2 \) \( \frac{1}{2} \) and for \( \lambda = 3 \) \( -1 \).

b) (10 points)

Find the solution, if it exists, of
\[
\begin{align*}
    x_1 + 2x_2 - 2x_3 + 3x_4 - 4x_5 &= -3 \\
    2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 &= -1 \\
    -x_1 - 2x_2 - 3x_4 + 11x_5 &= 15
\end{align*}
\]

Solution:
\[
\begin{bmatrix}
1 & 2 & -2 & 3 & -4 & -3 \\
2 & 4 & -5 & 6 & -5 & -1 \\
-1 & -2 & 0 & -3 & 11 & 15
\end{bmatrix}
\begin{bmatrix}
R_1 \\
R_2 \\
R_3
\end{bmatrix}
= 
\begin{bmatrix}
1 & 2 & -2 & 3 & -4 & -3 \\
0 & 0 & -1 & 0 & 3 & 5 \\
0 & 0 & -2 & 0 & 7 & 12
\end{bmatrix}
= 
\begin{bmatrix}
1 & 2 & -2 & 3 & -4 & -3 \\
0 & 0 & 1 & 0 & -3 & -5 \\
0 & 0 & 0 & 0 & 1 & 2
\end{bmatrix}
\]

Since the rank of the coefficient matrix equals the rank of the augmented matrix, there exists a solution. It is

\[
x_5 = 2 \quad x_3 - 3x_5 = -5 \quad \text{or} \quad x_3 = -5 + 3x_3 = -5 + 6 = 1 \quad \text{and} \quad x_1 = -2x_2 - 3x_4 + 7
\]

**Problem 4**

a) (13 points)

Verify Green’s theorem when \( P = 4x - 2y \); \( Q = 2x + 6y \) and \( C \) is the ellipse \( x = 2\cos \theta, \ y = \sin \theta, \ 0 \leq \theta \leq 2\pi \). (Recall that the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is \( \pi ab \).)

**SOLUTION**

For this ellipse, \( a = 2 \) and \( b = 1 \). Let \( G \) be the interior of \( C \). Green’s theorem states that the two integrals \( \int_C Pdx + Qdy \) and \( \iint_G (Q_y - P_x)dx\,dy \) are equal. We must verify this.

Since \( Q_x = 2 \) and \( P_y = -2 \),
\[
\iint_G (Q_x - P_y)dx\,dy = \iint_G 4dx\,dy = 4\iint_G dx\,dy = 4(\text{Area of } G) = 4(\pi)(2)(1) = 8\pi
\]

The ellipse is already parametrized by \( \theta \). Since \( dx = -2\sin \theta d\theta \) and \( dy = \cos \theta d\theta \),
\[
\int_C Pdx + Qdy = \int_C (4x - 2y)dx + (2x + 6y)dy
= \int_0^{2\pi} ((8\cos \theta - 2\sin \theta)(-2\sin \theta) + (4\cos \theta + 6\sin \theta)(\cos \theta))d\theta
= \int_0^{2\pi} (-16\sin \theta \cos \theta + 4\sin^2 \theta + 4\cos^2 \theta + 6\sin \theta \cos \theta) d\theta
= \int_0^{2\pi} (4 - 10\sin \theta \cos \theta) d\theta = 8\pi
\]

The theorem has now been verified.

**Problem 4**
b) (12 points)
Consider \( \int_0^2 \int_y^2 f(x,y) \, dx \, dy \).

a) Sketch the region of integration.
b) Write the integral reversing the order of integration.
c) Rewrite the integral in terms of polar coordinates.

**SOLUTION**

b) Taking the limits from the sketch, we get

\[
\int_0^2 \int_0^2 f(x,y) \, dy \, dx
\]

c) The limits on \( \theta \) are clear from the sketch. Noting that the polar equation of the line \( x = 2 \) is \( r \cos \theta = 2 \) or \( r = 2 \sec \theta \), we have

\[
\int_0^{\pi/4} \int_0^{2 \sec \theta} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.
\]

Don’t forget the extra factor of \( r \) inside the integral.

**Problem 5**
Consider the \( \int_C \vec{F} \cdot d\vec{r} \), where

\[
\vec{F} = (2xyz + z^2y) \hat{i} + (x^2z + z^2x) \hat{j} + (x^2y + 2xyz) \hat{k}
\]

a) (12 points)
Show that \( \nabla \times \vec{F} = \vec{0} \). What does this tell you about \( \int_C \vec{F} \cdot d\vec{r} \), where \( C \) is any closed curve?

**SOLUTION**

\[
\nabla \times \vec{F} = \text{curl } \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2xyz + z^2y & x^2z + z^2x & x^2y + 2xyz
\end{vmatrix}
\]

\[
= (x^2 + 2xz - x^2 - 2xz) \hat{i} - (2xy + 2yz - 2xy - 2zy) \hat{j} + (2xz + z^2 - 2xz - z^2) \hat{k} = \vec{0}
\]

Then \( \int_C \vec{F} \cdot d\vec{r} = 0 \) for any closed curve \( C \).

Or, equivalently, \( \int_C \vec{F} \cdot d\vec{r} \) is independent of the path taken between two given points.
b) (13 points)
Find a function $\Phi(x,y,z)$ such that $\nabla \Phi = \bar{F}

SOLUTION
$\bar{F} = \nabla \Phi = \frac{\partial \Phi}{\partial x} \hat{i} + \frac{\partial \Phi}{\partial y} \hat{j} + \frac{\partial \Phi}{\partial z} \hat{k}$

We set equal the corresponding components.

$$\frac{\partial \Phi}{\partial x} = 2xyz + z^2y \quad \Rightarrow \quad \Phi(x,y,z) = x^2yz + xyz^2 + C_1(y,z)$$

$$\frac{\partial \Phi}{\partial y} = x^2z + z^2x \quad \Rightarrow \quad \Phi(x,y,z) = x^2yz + xyz^2 + C_2(x,z)$$

$$\frac{\partial \Phi}{\partial z} = x^2y + 2xyz \quad \Rightarrow \quad \Phi(x,y,z) = x^2yz + xyz^2 + C_3(x,y)$$

Comparing the three expression for $\Phi$, we let $C_1 = C_2 = C_3 = C$. Since $C_1$ is independent of $x$, so is $C$. Likewise, $C_2$ is independent of $y$, and $C_3$ of $z$. Therefore $C$ is independent of all variables, i.e. it is a constant.

Finally, we have $\Phi(x,y,z) = x^2yz + xyz^2 + C$

Problem 6
a) (12 points)
Let $S$ be the closed surface bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, $y = z = 2$. Sketch $S$.

SOLUTION

b) (13 points)
Let $S$ be the closed surface in 6 a), $\vec{n}$ the outward unit normal to $S$, and $\bar{F} = xy\hat{i} + (y^2 + e^{z^2})\hat{j} + \sin xy\hat{k}$. Use the Divergence Theorem to transform the $\iiint_S \bar{F} \cdot \vec{n} dS$ into a triple integral. Do not evaluate the integral.
SOLUTION

\[ \iiint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \text{div} \vec{F} \, dV = \iiint_V (y + 2y + 0) \, dV = \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} 3y \, dy \, dz \, dx \]

The limits can be deduced from the sketch. Other correct expressions include:

\[ \int_{0}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{0}^{2-z} 3y \, dy \, dz \]

\[ \int_{0}^{1} \int_{0}^{2-z} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 3y \, dx \, dy \, dz \]

There is no way to express the integral in the forms \( dzdydx \) or \( dzdxdy \) or \( dxdzdy \) without splitting the integral into two smaller integrals.

**Problem 7**

a) (10 points)

Without expanding show that

\[
\begin{vmatrix}
1 & a & b + c \\
1 & b & c + a \\
1 & c & a + b
\end{vmatrix} = 0
\]

**SOLUTION:**

Using elementary operations on the columns we have,

\[
\begin{vmatrix}
1 & a & b + c \\
1 & b & c + a \\
1 & c & a + b
\end{vmatrix} \rightarrow -C_2 + C_1
\]

\[
\begin{vmatrix}
1 & 0 & b + c \\
1 & b - a & c + a \\
1 & c - a & a + b
\end{vmatrix} \rightarrow C_2 + C_3
\]

\[
\begin{vmatrix}
1 & 0 & b + c \\
1 & b - a & b + c \\
1 & c - a & b + c
\end{vmatrix}
\]

\[
= (b + c) \begin{vmatrix}
1 & 0 & 1 \\
1 & b - a & 1 \\
1 & c - a & 1
\end{vmatrix} = 0
\]

Since the first and third columns are the same.

b) (15 points)

Use Stokes’ Theorem to compute the integral \( \oiint_S \text{curl} \vec{F} \cdot \vec{n} \, dS \), where \( \vec{F} = yzi + xzj + xyk \), and \( S \) is the part of sphere \( x^2 + y^2 + z^2 = 4 \) that lies inside the cylinder \( x^2 + y^2 = 1 \) and above the \( xy \)-plane. Sketch \( S \).

(Note: \( \cos^2 t - \sin^2 t = \cos 2t \).)

**SOLUTION:**

Stoke’s Theorem states that \( \oiint_S \text{curl} \vec{F} \cdot \vec{n} \, dS = \int_{\partial S} \vec{F} \cdot d\vec{r} \). We want to find \( \int_{\partial S} \vec{F} \cdot d\vec{r} \).
The region $S$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and is above the $xy$-plane. To find where the sphere and the cylinder intersect we set $x^2 + y^2 = 1$ in the equation $x^2 + y^2 + z^2 = 4$. This yields $1 + z^2 = 4$ or $z = \sqrt{3}$. Thus $\partial S$ is given by $x^2 + y^2 = 1, z = \sqrt{3}$. We parametrize this as $x = \cos t, y = \sin t, z = \sqrt{3}$, $0 \leq t \leq 2\pi$.

Hence $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + \sqrt{3} \hat{k}$ and $\vec{r}'(t) = -\sin t \hat{i} + \cos t \hat{j} + 0 \hat{k}$ and $\vec{F}(t) = \sqrt{3} \sin t \hat{i} + \sqrt{3} \cos t \hat{j} + \cos t \sin t \hat{k}$

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{0}^{2\pi} \vec{F}(t) \cdot \vec{r}'(t) \, dt = \int_{0}^{2\pi} \sqrt{3} (\cos t - \sin t) \, dt = \sqrt{3} \int_{0}^{2\pi} \cos 2t \, dt = 0$$

Problem 8

a) (10 points)
Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$. Sketch the volume.

SOLUTION:

The paraboloid $z = 1 - x^2 - y^2$ intersects the $x,y$-plane on the circle $x^2 + y^2 = 1$. Let $D$ denote the inside of the circle. Then the volume is

$$V = \iiint_D 1 - x^2 - y^2 \, dz \, dA$$

Using cylindrical coordinates $x = r \cos \theta, \ y = r \sin \theta, \ z = z$ we have,

$$V = \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{1-r^2} r \, dz \, dr \, d\theta = \int_{0}^{2\pi} \int_{0}^{1} (1 - r^2) \, r \, dr \, d\theta = \frac{\pi}{2}$$

b) (15 points)
Find the eigenvalues and eigenfunctions of

$$y''' + \lambda y = 0 \quad y(-\pi) = y(\pi) \quad y'(-\pi) = y'(\pi)$$
SOLUTION:

We must consider the cases \( \lambda < 0, \lambda = 0, \lambda > 0 \).

Case I. \( \lambda < 0 \)

Let \( \lambda = -a^2 \)

\[
y'' - a^2 y = 0
\]

The auxiliary equation is \( r^2 - a^2 = 0 \) which tells us that \( r = \pm a \) real distinct roots.

The characteristic equation is

\[
y = c_1 e^{ax} + c_2 e^{-ax}
y' = ac_1 e^{ax} - ac_2 e^{-ax}
\]

\[
y(-\pi) = y(\pi) \Rightarrow c_1 e^{ax} + c_2 e^{-ax} = c_1 e^{ax} + c_2 e^{-ax} \Rightarrow c_1(e^{ax} - e^{-ax}) = c_2(e^{ax} - e^{-ax})
\]

\[
\Rightarrow c_1 = c_2
\]

\[
y'(-\pi) = y'(\pi) \Rightarrow ac_1 e^{-ax} - ac_2 e^{ax} = ac_1 e^{ax} - ac_2 e^{-ax}
\]

But \( c_1 = c_2 \) so \( 2c_1 e^{ax} = 2c_1 e^{-ax} \). \( \Rightarrow c_1 = 0 \) so that \( c_2 = 0 \). Thus \( y = 0 \) is the only solution, and there are no negative eigenvalues.

Case II. \( \lambda = 0 \)

\[
y'' = 0
\]

The auxiliary equation is \( r^2 = 0 \), a repeated root.

\[
y = c_1 x + c_2
\]

\[
y(-\pi) = y(\pi) \Rightarrow -c_1 \pi + c_2 = c_1 \pi + c_2 = 0
\]

This implies that \( c_1 = 0 \). This also satisfies \( y'(-\pi) = y'(\pi) \) so \( y = c_2 \) where \( c_2 \neq 0 \) is a nontrivial solution corresponding to \( \lambda = 0 \).

Case III. \( \lambda > 0 \)

Let \( \lambda = a^2 \), where \( a \neq 0 \). The DE is then

\[
y'' + a^2 y = 0
\]

The auxiliary equation is \( r^2 + a^2 = 0 \) which tells us that \( r = \pm ai \) complex roots.

The characteristic equation is

\[
y = c_1 \cos ax + c_2 \sin ax
\]

\[
y' = -c_1 a \sin ax + c_2 a \cos ax
\]

\[
y(-\pi) = y(\pi) = 0 \Rightarrow c_1 \cos(-a\pi) + c_2 \sin(-a\pi) = c_1 \cos a\pi + c_2 \sin a\pi \text{ or}
\]

\[
c_1 \cos(a\pi) - c_2 \sin(a\pi) = c_1 \cos a\pi + c_2 a \sin a\pi \Rightarrow 2c_2 \sin a\pi = 0
\]

\[
y'(-\pi) = y'(\pi) \text{ implies that } -c_1 a \sin(-a\pi) + c_2 a \cos(-a\pi) = -c_1 \sin a\pi + c_2 \cos a\pi
\]

\[
\Rightarrow 2c_1 a \sin a\pi = 0. \text{ Thus if } \sin a\pi \neq 0, \text{ since we have } c_1 = c_2 = 0. \text{ Hence for a nonzero solution we must have } \sin a\pi = 0.
\]

This is true when \( a = n \), where \( n = \pm 1, \pm 2, \pm 3, \ldots \)

Therefore the eigenvalues are \( \lambda_n = n^2 \) and the eigenfunctions are

\[
y_n = a_n \cos n\pi x + b_n \sin n\pi x.
\]
Problem 9

a) (13 points) Suppose that \( f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x) \), where \( \{\phi_1, \phi_2, \phi_3, \ldots\} \) is an orthonormal set on the interval \([a, b]\). Show that \( \int_{a}^{b} f^2(x) \, dx = \sum_{i=1}^{\infty} a_i^2 \)

SOLUTION:

By definition of orthonormality we know that

\[
\int_{a}^{b} \phi_i(x)\phi_j(x) \, dx = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\]

\[
\int_{a}^{b} f^2(x) \, dx = \langle f, f \rangle = \sum_{i=1}^{\infty} a_i \langle \phi_i(x), \phi_i(x) \rangle > \sum_{i=1}^{\infty} a_i \sum_{j=1}^{\infty} a_j \langle \phi_j(x), \phi_j(x) \rangle >
\]

\[
= a_1^2 < \phi_1(x), \phi_1(x) > + a_1 a_2 < \phi_1(x), \phi_2(x) > + a_1 a_3 < \phi_1(x), \phi_3(x) > + \cdots + a_2 a_1 < \phi_2(x), \phi_1(x) > + a_2^2 < \phi_2(x), \phi_2(x) > + \cdots
\]

However, since we have an orthonormal set we have that

\[
\int_{a}^{b} f^2(x) \, dx = a_1^2 < \phi_1(x), \phi_1(x) > + a_2^2 < \phi_2(x), \phi_2(x) > + \cdots = \sum_{i=1}^{\infty} a_i^2
\]

b) (12 points)

Find the inverse of the matrix

\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
2 & 3 & 0
\end{bmatrix}
\]

SOLUTION:

Using row reduction,

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
2 & 3 & 0 & 0 & 0 & 1
\end{bmatrix} \rightarrow R_2 \leftrightarrow R_1 \\
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix} \rightarrow -2R_1 + R_2 \\
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 6 & 2 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & -6 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix} \rightarrow -6R_3 + R_2 \\
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & -6 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix} \rightarrow -2R_2 + R_1 \\
\begin{bmatrix}
1 & 0 & 3 & -3 & 6 & -2 \\
0 & 1 & 0 & 2 & -6 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 3 & -3 & 6 & -2 \\
0 & 1 & 0 & 2 & -6 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]
$$\rightarrow^{-3R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 & -3 & -9 & -2 \\ 0 & 1 & 0 & 2 & -6 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The inverse is

$$\begin{bmatrix} -3 & 9 & 2 \\ 2 & -6 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$