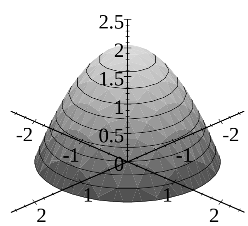
Ma 227 Exam III Solutions Spring 1998

1 a) Use cylindrical coordinates to parametrize the surface $z = 2 - x^2 - y^2$, $z \geq 0$

The transforming equations for cylindrical coordinates are:
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
This is not a parametrization for any surface because x, y, and z depend on *three* other variables. However, we can eliminate z from the right-hand side of the equations by noting that $z = 2 - r^2$ at every point on the surface. Then the equations:
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = 2 - r^2$$
form a parametrization for $z = 2 - x^2 - y^2$ in terms of the *two* parameters r and θ. The limits $0 \leq r \leq \sqrt{2}$ and $0 \leq \theta \leq 2\pi$ can be gotten from a sketch of the surface. (See below.)

1 b) Use the parametrization in cylindrical coordinates from 1 a) to write an iterated integral for the surface area of the surface given in a). Do *not* evaluate this integral.



Use the equations from $1a)$ and recall that $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

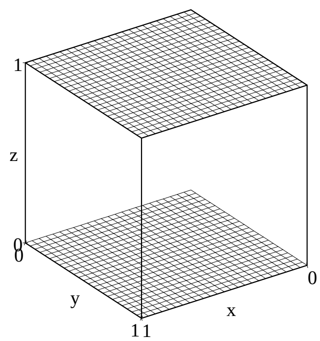
Then we have $\vec{r} = r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath} + \left(2 - r^2\right)\hat{k}$ and $\vec{r}_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} - 2r\hat{k}$ and $\vec{r}_\theta = -r\sin\theta\hat{\imath} + r\cos\theta\hat{\jmath}$

Next calculate the cross product: $$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = 2r^2\cos\theta\hat{\imath} + 2r^2\sin\theta\hat{\jmath} + r\hat{k}$$

Then the surface element $dS = |\vec{r}_r \times \vec{r}_\theta| \, drd\theta = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}\, drd\theta = r\sqrt{4r^2+1}\, drd\theta$
Since this surface *uniquely* projects onto the *xy* plane, you can also use the formula
$dS = \sqrt{1 + z_x^2 + z_y^2}\, dxdy = \sqrt{1 + 4x^2 + 4y^2}\, dxdy = r\sqrt{4r^2+1}\, drd\theta$

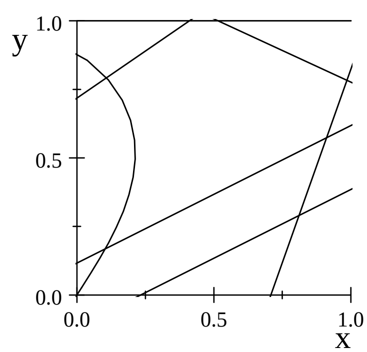
Therefore the surface area is $\iint_S dS = \int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{4r^2+1}\, drd\theta$

2 a) Let S be the closed surface of the cubic box bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1.$ Let $\vec{v} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$. Use the Divergence Theorem to find $\iint_S \vec{v} \cdot \vec{n} dS$.



By the Divergence Theorem $\iint_S \vec{v} \cdot \vec{n} dS = \iiint_V \operatorname{div} \vec{v}\, dV$
$= \iiint_V (4z - 2y + y)dxdydz = \int_0^1\int_0^1\int_0^1 (4z - y)dzdydx = \int_0^1\int_0^1 (2 - y)dydx = \int_0^1 \frac{3}{2}dx = \frac{3}{2}$

2 b) Let S^* be the above box which is now open at the bottom, i.e., at the end $z = 0$. Use Stokes' Theorem to show that $\iint_{S^*} \left(curl\, \vec{v}\right) \cdot \vec{n} dS = 0$. Here $\vec{n}$ is the outward unit normal.



This is ∂S^*, the boundary of the surface S^*. It is a closed curve made up of four line segments. On ∂S^*, $z = 0$.

By Stokes' Theorem: $\iint_{S^*} \left(curl\, \vec{v}\right) \cdot \vec{n} dS = \oint_{\partial S^*} \vec{v} \cdot d\vec{r} = \oint_{\partial S^*} 4xzdx - y^2dy + yzdz = \oint_{\partial S^*} -y^2dy$
using the fact that $z = 0$ on ∂S^*. Our task is therefore to show that the line integral $\oint_{\partial S^*} -y^2dy = 0$.

Notice that y is a constant on the line segments $y = 0$ and $y = 1$. Then $dy = 0$ on those segments. Then we only need to show that the line integrals on the other two pieces add up to zero.
$\oint_{\partial S^*} -y^2 dy = \int 0 + \int_0^1 -y^2 dy + \int 0 + \int_1^0 -y^2 dy = \int_0^1 -y^2 dy - \int_0^1 -y^2 dy = 0$. Therefore it follows $\iint_{S^*} \left(curl\, \vec{v} \right) \cdot \vec{n} dS = 0$.

3 a) Find the inverse of $\begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$.

Adjoin I to the matrix and row reduce until the original matrix becomes I.

$\begin{bmatrix} 3 & 2 & 1 & 0 \\ -2 & -2 & 0 & 1 \end{bmatrix}$ First, add row 2 to row 1 to produce a 1 in the 1,1 position.

$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -2 & -2 & 0 & 1 \end{bmatrix}$ Next, add twice row 1 to row 2 to get a zero in the 2,1 position.

$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -2 & 2 & 3 \end{bmatrix}$ Finally, divide R2 by -2 to get a 1 in the 2,2 position. There is already a zero in the 1,2 position. This yields:

$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -\frac{3}{2} \end{bmatrix}$ Therefore the desired inverse is $\begin{bmatrix} 1 & 1 \\ -1 & -\frac{3}{2} \end{bmatrix}$

3 b) What is the rank of $\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & 6 & -1 & 5 \\ -2 & 0 & 6 & 8 \\ -1 & 5 & 9 & 15 \end{bmatrix}$?

We must put this matrix into row-reduced echelon form, then count the rows which are not all zero.

$\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & 6 & -1 & 5 \\ -2 & 0 & 6 & 8 \\ -1 & 5 & 9 & 15 \end{bmatrix}$ First, add R1 to R4 and add 2*R1 to R3. This gives three zeroes in the first column.

$\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & 6 & -1 & 5 \\ 0 & -2 & 14 & 12 \\ 0 & 4 & 13 & 17 \end{bmatrix}$ Next, since -2 is easier to manipulate than 6, switch R2 and R3.

$\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & -2 & 14 & 12 \\ 0 & 6 & -1 & 5 \\ 0 & 4 & 13 & 17 \end{bmatrix}$ Now, add 3*R2 to R3 and add 2*R2 to R4. This gives two zeroes in the second column.

$\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & -2 & 14 & 12 \\ 0 & 0 & 41 & 41 \\ 0 & 0 & 41 & 41 \end{bmatrix}$ Finally, add -1*R3 to R4 to produce $\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & -2 & 14 & 12 \\ 0 & 0 & 41 & 41 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This last matrix is in row-reduced echelon form because *each row has more leading zeroes than the one above it*. Therefore the rank is 3.

3 c) Does the system $\begin{array}{rrrcl} x_1 & -x_2 & +4x_3 & = & 2 \\ & 6x_2 & -x_3 & = & 5 \\ -2x_1 & & +6x_3 & = & 8 \\ -x_1 & +5x_2 & +9x_3 & = & 15 \end{array}$ possess a solution or solutions? If it does, then find it (them).

The augmented matrix associated with the system is $\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & 6 & -1 & 5 \\ -2 & 0 & 6 & 8 \\ -1 & 5 & 9 & 15 \end{bmatrix}$, which is the same as the matrix in question 3 b). Therefore it reduced to the matrix $\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & -2 & 14 & 12 \\ 0 & 0 & 41 & 41 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Dividing R2 by -2 and R3 by 41 we get $\begin{bmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & -7 & -6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The rank of the augmented matrix we already found to be 3. The rank of the coefficient matrix (i.e. not including the last column) is also three. Since the ranks are equal, the system will have a solution. To find it, we can convert the last matrix back to a system of equations:

$$\begin{aligned} x_1 - x_2 + 4x_3 &= 2 \\ x_2 - 7x_3 &= -6 \\ x_3 &= 1 \end{aligned}$$

It is easy to see that $x_1 = -1$, $x_2 = 1$, and $x_3 = 1$.