Ma 227 Line Integrals

Definition. Let $P(x,y)$ and $Q(x,y)$ be functions of two variables whose first partial derivatives are continuous in an open rectangle H of the x,y – plane. Consider an arc (curve) C in H whose parametric equations are

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$$

and are such that as t increases from a to b, the corresponding point $(f(t),g(t))$, traces the arc C from the point $A = (f(a),g(a))$ to the point $B = (f(b),g(b))$. Let f' and g' be continuous for $a \leq t \leq b$.
Then

$$\int_C P(x,y)dx + Q(x,y)dy = \int_a^b \{P(f(t),g(t))f'(t) + Q(f(t),g(t))g'(t)\}dt$$

is called the line integral of $P(x,y)dx + Q(x,y)dy$ along C from A to B.
Remark: Notice that the right hand side above is an ordinary definite integral.

Example: Evaluate the line integral

$$\int_C (x^2 - y^2)dx + 2xydy$$

along the curve C whose parametric equations are

$$x = t^2; \qquad y = t^3; \qquad 0 \leq t \leq \frac{3}{2}$$

Solution: $f(t) = t^2$ and $g(t) = t^2$. $\Rightarrow$ $f' = 2t$ and $g' = 3t^2$.

$$\int_C (x^2 - y^2)dx + 2xydy = \int_0^{\frac{3}{2}} [(t^4 - t^6)(2t) + 2t^2t^3(3t^2)]dt$$

$$= \int_0^{\frac{3}{2}} [(2t^5 + 4t^7)dt = \frac{8505}{512}$$

Remark: C may be described vectorially via

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$$

$\Rightarrow$

$$\vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j}$$

If we let

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j},$$

then

$$\vec{F}(t) = \vec{F}(f(t),g(t)) = P(f(t),g(t))\vec{i} + Q(f(t),g(t))\vec{j}$$

$\Rightarrow$

$$\vec{F}(f(t),g(t)) \cdot \vec{r}\,'(t) = P(f(t),g(t))f'(t) + Q(f(t),g(t))g'(t)$$

Hence

$$\int_C [P(x,y)dx + Q(x,y)dy] = \int_a^b \vec{F}(f(t)) \underbrace{\cdot \vec{r}\,'(t)dt}_{d\vec{r}} = \int_C \vec{F} \cdot d\vec{r}$$

Remark: The results we have given for two dimensions readily go over to three dimensions. We define the three dimensional line integral as follows:

The curve may be described in three dimensions via

$$x = f(t); \qquad y = g(t); \qquad z = h(t)$$

or

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

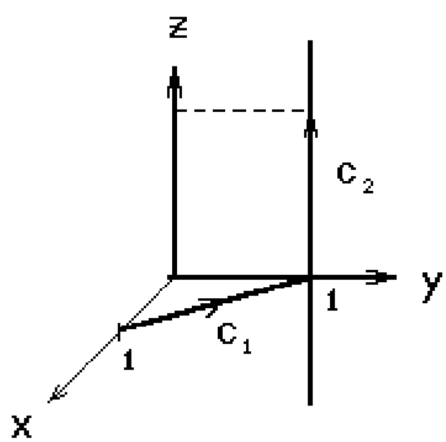
If

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz = \int_a^b \vec{F}(f(t), g(t), h(t)) \cdot \vec{r}\,'(t)dt$$
$$= \int_a^b \{P(f(t), g(t), h(t))f'(t) + Q(f(t), g(t), h(t))g'(t) + R(f(t), g(t), h(t))h'(t)\}dt$$

Example: Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = xy\vec{i} + xz\vec{j} - y\vec{k}$ and C is the directed line segment C_1 from $(1,0,0)$ to $(0,1,0)$ followed by C_2 which is the segment from $(0,1,0)$ to $(0,1,1)$.



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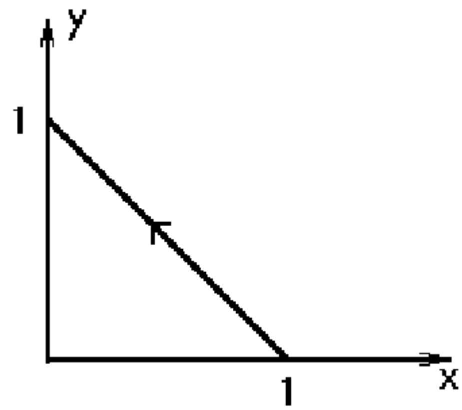

Solution: On C_1 $\quad z = 0$

$$y = -x+1 \qquad or x = 1-y$$

Let $y = t \quad x = 1-t \quad 0 \le t \le 1$

$$\vec{r}(t) = (1-t)\vec{i} + t\vec{j} + 0 \cdot \vec{k} \Rightarrow \vec{r}'(t) = -\vec{i} + \vec{j}$$

$$\vec{F} = xy\vec{i} + xz\vec{j} - y\vec{k} \Rightarrow \vec{F}(t) = (1-t)t\vec{i} + 0\vec{j} - t\vec{k}$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t) \cdot r'(t)dt = \int_0^1 [t^2 - t]dt = -\frac{1}{6}$$

On C_2 $\quad x = 0, \quad y = 1, \quad z$ goes from 0 to1

Let $z = t \quad 0 \le t \le 1 \quad \Rightarrow \vec{r}(t) = 0\vec{i} + \vec{j} + t\vec{k}; \quad \vec{F} = 0\vec{i} + 0\vec{j} - \vec{k}$ and $\vec{r}'(t) = \vec{k}$

$$\int_{C_2} = \int_0^1 -dt = -1.$$

$\Rightarrow$

$$\int_C = \int_{C_1} + \int_{C_2} = -\frac{1}{6} - 1 = -\frac{7}{6}$$

$$\int_C fds$$

Two Dimensions

Let C denote a plane curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

or equivalently by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Assume the curve is **smooth**, which means that the tangent vector $\mathbf{r}' = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ is continuous and never the zero vector. Let $f(x,y)$ be a function defined at each point of the curve C. The line integral of f along C is defined by the formula

$$\int_C f(x,y)ds = \lim_{\|P\|\to 0} \sum_{i=1} f(x_i,y_i)\Delta s_i$$

In this formula s denotes arc length along the curve, P denotes a partition of the curve into n pieces, and $\|P\|$ is the length of the longest piece. (x_i,y_i) is a point on the i^{th} piece. (The definition of an ordinary integral $\int_a^b g(x)dx$ is defined by a special case of this process, in which the curve C is the segment of the x-axis between a and b.) In practice, this limiting process is rarely carried out, since

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\frac{ds}{dt}dt$$

and the integral on the right is an ordinary integral. Recall that $ds = \sqrt{dx^2+dy^2}$ and hence

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Thus

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example Evaluate

$$\int_C (2+x^2y)ds$$

where C is the upper half of the unit circle $x^2+y^2 = 1$.

Solution: We parametrize the upper half of the unit circle using

$$x = \cos t \quad y = \sin t \quad 0 \le t \le \pi$$

Then

$$\begin{aligned}\int_C (2+x^2y)ds &= \int_0^\pi (2+\cos^2 t\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_0^\pi (2+\cos^2 t\sin t)\sqrt{\sin^2 t+\cos^2 t}\,dt \\ &= \int_0^\pi (2+\cos^2 t\sin t)dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi \\ &= 2\pi + \frac{2}{3}\end{aligned}$$

Three Dimensions

Let C denote a space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

or equivalently by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Assume the curve is **smooth**, which means that the tangent vector $\mathbf{r}' = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{j}$ is continuous and never the zero vector. Let $f(x,y,z)$ be a function defined at each point of the curve C. The line integral of f along C is defined by the formula

$$\int_C f(x,y,z)ds = \lim_{\|P\|\to 0} \sum_{i=1} f(x_i,y_i,z_i)\Delta s_i.$$

Here, s denotes arc length along the curve, P denotes a partition of the curve into n pieces, and $\|P\|$ is the length of the longest piece. The point (x_i,y_i,z_i) is a point on the i^{th} piece. In practice, this limiting process is rarely carried out, since

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\frac{ds}{dt}dt$$

and the integral on the right is an ordinary integral. In this case $ds = \sqrt{dx^2+dy^2+dz^2}$ and therefore

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

Example

Evaluate

$$\int_C y\sin z\,ds$$

where C is the circular helix given by the equations

$$x = \cos t,\ y = \sin t\ \ z = t\ \ 0 \le t \le 2\pi$$

Solution:

$$\begin{aligned}\int_C y\sin z\,ds &= \int_0^{2\pi}(\sin t)\sin t\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt\\ &= \int_0^{2\pi}\sin^2 t\sqrt{\sin^2 t+\cos^2 t+1}\,dt\\ &= \frac{\sqrt{2}}{2}\int_0^{2\pi}(1-\cos 2t)dt\\ &= \frac{\sqrt{2}}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi} = \sqrt{2}\,\pi\end{aligned}$$

Path Independence

Find the value of

$$\int_C y^2dx+(x-y)dy$$

from the point $A = (0,-2)$ to the point $B = (28,6)$

(a) along the path $x = t^3+1; \quad y = 2t; \quad -1 \le t \le 3;$

(b) along the straight line segment AB

Solution:

(a) first $x = t^3+1 \quad y = 2t \Rightarrow x = \frac{y^3}{8}+1$ or $y^3 = 8x-8$

$$\vec{F}(x,y) = y^2\vec{i}+(x-y)\vec{j} \qquad \vec{r} = (t^3+1)\vec{i}+2t\vec{j}$$

$$\vec{F}(t) = (2t)^2\vec{i}+(t^3+1-2t)\vec{j} \qquad \vec{r}'(t) = 3t^2\vec{i}+2\vec{j}$$

$$\int_C = \int_{-1}^3\{(2t)^2\cdot(3t^2)+(t^3-2t+1)\cdot 2\}dt = \frac{12t^5}{5}+\frac{2t^4}{4}-\frac{4t^2}{2}+2t\Big|_{-1}^3 = \frac{3088}{5}$$

Along path (b): Line goes from $(0,-2)$ to $(28,6)$

$\Rightarrow$ slope $m = \frac{6+2}{28} = \frac{2}{7} \Rightarrow y+2 = \frac{2}{7}x$ or $y = \frac{2}{7}x-2$

Let $x = \frac{7}{2}t \Rightarrow y = t-2 \qquad 0 \le t \le 8$

$$\vec{F}(t) = (t-2)^2\vec{i} + \left(\frac{7}{2}t - t + 2\right)\vec{j} = (t-2)^2\vec{i} + \left(\frac{5}{2}t + 2\right)\vec{j}$$

$$\vec{r}(t) = \frac{7}{2}t\vec{i} + (t-2)\vec{j} \Rightarrow \vec{r}\,'(t) = \frac{7}{2}\vec{i} + \vec{j}$$

$$\int_C = \int_0^8 \left\{\frac{7}{2}(t-2)^2 + \frac{5}{2}(t+2)\right\}dt = \frac{1072}{3}$$

Notice that the two paths give two different results.

Often one must consider situations in which the path C is a closed curve. Hence the starting point A and ending point B are the same. This is usually written as

$$\oint_C \vec{F} \cdot d\vec{r}.$$

For plane curves we take the positive direction of C so that the interior of the closed curve is always to the left as C is traversed.

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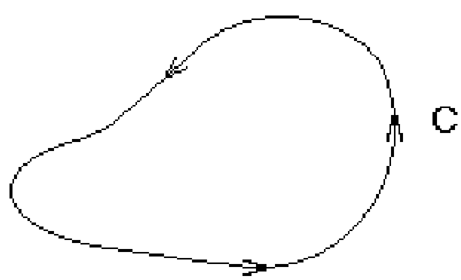


Example: Show that

$$\oint_C \frac{xdy - ydx}{x^2 + y^2} = 2\pi,$$

where C is the circle $x^2 + y^2 = a^2$

Solution: Let $x = a\cos t \qquad y = a\sin t \qquad 0 \le t \le 2\pi$

$$\oint_C = \int_0^{2\pi} \left\{\frac{a\cos t(a\cos t) - a\sin t(-a\sin t)}{a^2}\right\}dt$$
$$= \int_0^{2\pi} \{\cos^2 t + \sin^2 t\}dt = \int_0^{2\pi} dt = 2\pi$$

We have seen that the value of a line integral depends on the integrand, the endpoints A and B, and the arc C from A to B. However, certain line integrals depend only on the integrand and endpoints A and B. Such integrals are called path independent or are said to be independent of the path.

Example: Show that the value of the integral

$$\int_C (3x^2 - 6xy)dx + (-3x^2 + 4y + 1)dy$$

is independent of the path taken from $(-1,2)$ to $(4,3)$.
Solution: Here $P = 3x^2 - 6xy \qquad Q = -3x^2 + 4y + 1$
Suppose we could find a function $G(x,y)$ such that

$$G_x = P \qquad G_y = Q$$

Then

$$\int_C Pdx + Qdy = \int_C G_x dx + G_y dy = \int_C dG = G(4,3) - G(-1,2).$$

which is a number independent of the path C.
This means that we want $P\,dx + Q\,dy$ to be an exact differential. The condition for this is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Here $P_y = -6x = Q_x \Rightarrow$ such a G exists. Now

$$G_x = P = 3x^2 - 6xy$$

$\Rightarrow$

$$G = x^3 - 3x^2y + g(y)$$

where $g(y)$ is a function of y.
But

$$G_y = -3x^2 + g'(y) = Q = -3x^2 + 4y + 1$$

$\Rightarrow$

$$g'(y) = 4y + 1 \qquad \text{or} \quad g(y) = 2y^2 + y + K.$$

Thus

$$G(x,y) = x^3 - 3x^2y + 2y^2 + y + K$$

where$C$ is a constant. Then $G(4,3) = -59 + K$ and $G(-1,2) = 3 + K$
Thus

$$\int_C (3x^2 - 6xy)dx + (-3x^2 + 4y + 1)dy = -59 + K - 3 - K = -62$$

We may summarize the above as follows:

Let $P(x,y)dx + Q(x,y)dy$ be an exact differential of some function G in an open rectangular region H. If C is an arc lying entirely in H with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \le t \le b$$

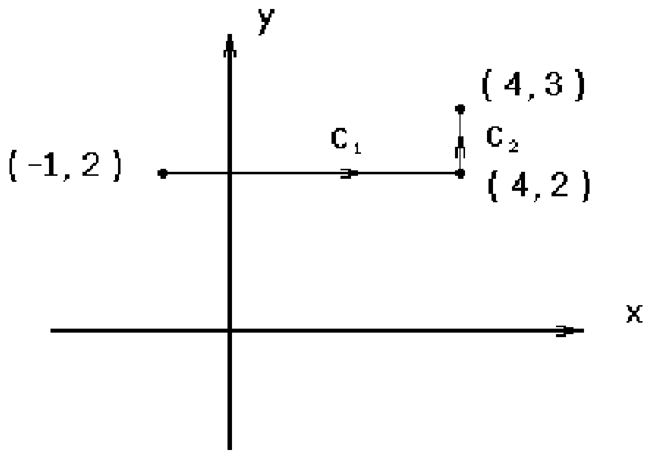
and f' and g' are continuous, then

$$\int_C P(x,y)dx + Q(x,y)dy = G(f(b),g(b)) - G(f(a),g(a))$$

where $(f(a),g(a))$ and $(f(b),g(b))$ are the endpoints of C.
Remark: If a line integral is path independent one may choose a path along which it is easy to evaluate the line

integral.

Example: $\int_C (3x^2 - 6xy)dx + (-3x^2 + 4y + 1)dy$ from $(-1,2))$ to $(4,3)$. (This is the same example we dealt with above.)



$$\int_C = \int_{C_1} + \int_{C_2}$$

Note that $dy = 0$ and $y = 2$ on C_1 and $dx = 0$ and $x = 4$ onC_2 $\Rightarrow$

$$\int_C = \int_{-1}^{4} (3x^2 - 6xy)dx + \int_2^3 (-3x^2 + 4y + 1)dy$$

But $y = 2$ in the first integral whereas $x = 4$ in the second$\Rightarrow$

$\int_C = \int_{-1}^{4} (3x^2 - 12x)dx + \int_2^3 (4y - 47)dy = -62$

Remark: Recall that

$$\nabla G = G_x \vec{i} + G_y \vec{j}$$

Also $d\vec{r} = dx\vec{i} + dy\vec{j}$ $\Rightarrow$

$$\nabla G \cdot d\vec{r} = G_x dx + G_y dy.$$

Therefore if $Pdx + Qdy$ is an exact differential, then

$$\int_C Pdx + Qdy = \int_C \nabla G(x,y) \cdot d\vec{r}$$

Remarks:

(1) The fact that a line integral is independent of path is equivalent to the statement that the line integral around any closed path is zero. To see this let C be any closed path and $P_0 \neq P_1$ be points on C.

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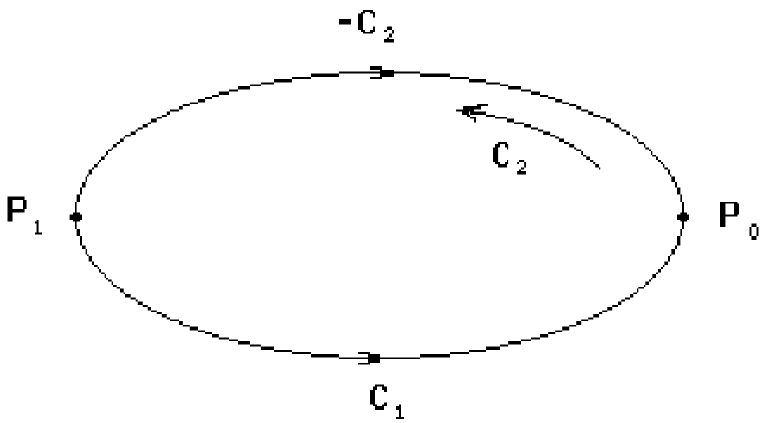


Then $C = C_1 + C_2$. If the line integral is path independent ⇒

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-C_2} \vec{F} \cdot d\vec{r}.$$

$$\text{Thus} \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

But $-d\vec{r}$ along $-C_2$ is equivalent to $d\vec{r}$ along C_2. Therefore

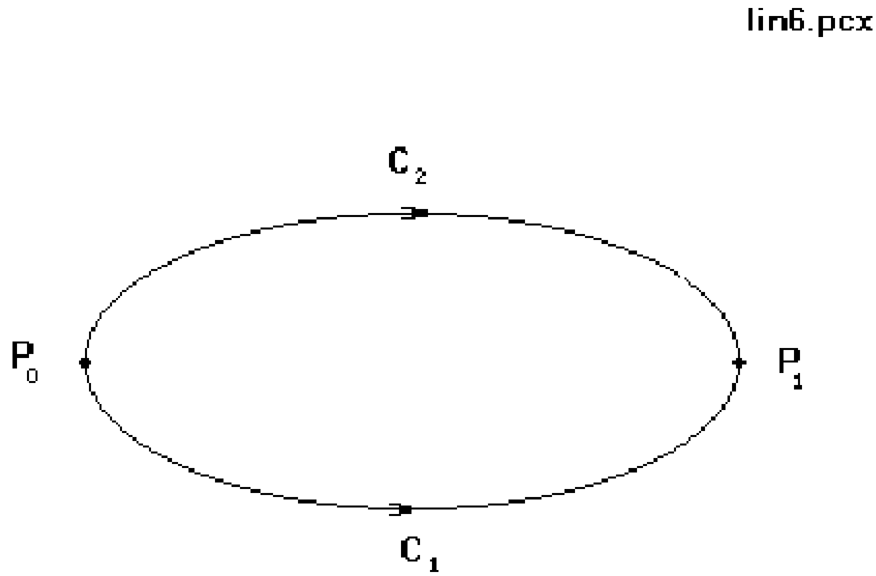
$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \oint_{C} \vec{F} \cdot d\vec{r} = 0$$

Suppose now that

$$\oint_{C} \vec{F} \cdot d\vec{r} = 0$$

for any closed path C.

Let P_0 and P_1 be any two points on C and C_1 and C_2 any two paths joining them.



Then $C = C_1 + (-C_2)$ is a closed path and

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1 - C_2} \vec{F} \cdot d\vec{r} = 0$$

$\Rightarrow$

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = 0 \quad \text{or} \quad \int_{C_1} \vec{F} \cdot d\vec{r} = -\int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

Hence the following are equivalent:

$\int_C \vec{F} \cdot d\vec{r}$ is path independent $\leftrightarrow$ there exists a G such that $\vec{F} = \nabla G$

$\leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed path C.

We have discussed path independence in two dimensions. Similar things hold in three dimensions.

Example: If $\vec{F} = y\vec{i} - z\vec{j} + x\vec{k}$ is $\int_C \vec{F} \cdot d\vec{r}$ path independent?

Solution: The line integral is path independent $\Leftrightarrow$ there exists a function $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}$. Suppose such a ϕ exists.

$\Rightarrow$

$$\phi_x = y; \qquad \phi_y = -z; \qquad \phi_z = x$$

Now $\phi_z = x \Rightarrow \phi(x,y,z) = xz + g(x,y)$ But $\quad \phi_x = z + \dfrac{\partial g}{\partial x} = y$

$\Rightarrow$

$$z = y - \frac{\partial g}{\partial x}(x,y)$$

But z is an independent variable and therefore not dependent upon x and y. Thus no such ϕ can exist $\Rightarrow \int_C \vec{F} \cdot d\vec{r}$ is path dependent for this $\vec{F}$.

Question: When does there exist a $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}$?

Theorem: Suppose $\vec{F}$ is a continuously differentiable function in a region D in space and that

$$curl\vec{F} = 0 \text{ in } D$$

Then there exists a continuously differentiable, scalar function $\phi(x,y,z)$, in D such that

$$\vec{F} = \nabla\phi.$$

Remark: $C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad a \le t \le b$. $\vec{F}$ force on a particle.
Then

$$Work = \int_C \vec{F} \cdot d\vec{r}$$

Example It can be shown that for the vector field

$$\vec{F}(x,y,z) = yz(2x+y)\vec{i} + xz(x+2y)\vec{j} + xy(x+y)\vec{k}$$

$$curl\vec{F} = \nabla \times \vec{F} = 0$$

Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where C is the curve given by the vector equation

$$\vec{r}(t) = (1+t)\vec{i} + (1+2t^2)\vec{j} + (1+3t^2)\vec{k} \qquad 0 \le t \le 1$$

Solution:

Check (not required)

$$\begin{aligned} &\nabla \times (yz(2x+y), xz(x+2y), xy(x+y)) \\ &= (x(x+y) + xy - x(x+2y), y(2x+y) - y(x+y) - xy, z(x+2y) + xz - z(2x+y) - yz) \\ &= (0,0,0) \end{aligned}$$

Thus there exists $f(x,y,z)$ such that $gradf = \vec{F}$.

$$f_x = yz(2x+y)$$

so

$$f = x^2yz + xy^2z + g(y,z)$$

Then

$$f_y = x^2z + 2xyz + g_y = xz(x+2y)$$

Therefore $g_y = 0$, so $g = h(z)$ and

$$f = x^2yz + xy^2z + h(z)$$

Also

$$f_z = x^2y + xy^2 + h'(z) = xy(x+y)$$

so $h'(z) = 0$. Thus

$$f(x,y,z) = x^2yz + xy^2z + K$$

$$\vec{r}(0) = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{r}(1) = 2\vec{i} + 3\vec{j} + 4\vec{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = f(2,3,4) - f(1,1,1) = 118$$

Green's Theorem

There is a remarkable theorem that identifies a double integral over a region *R* with a line integral around its boundary. It is known as Green's Theorem.

Theorem: Let $P(x,y)$ and $Q(x,y)$ be functions of two variables which are continuous and have continuous first partial derivatives in some rectangular region *H* in the x,y – plane. If *C* is a simple, closed, piecewise smooth curve lying entirely in *H*, and if *R* is the bounded region enclosed by *C*, then

$$\oint_C \{P(x,y)dx + Q(x,y)dy\} = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

Corollary: Let *R* be a bounded region in the x,y – plane. Then the area of *R* is given by

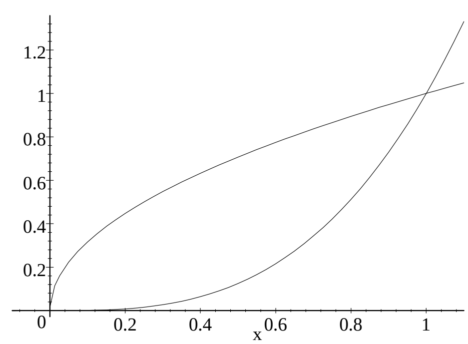
$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

where *C* is the boundary of *R*

Proof: Let $P = -\frac{y}{2}$ and $Q = \frac{x}{2}$ in Green's Theorem. ⇒

$$\oint_C \left(\frac{-y}{2}dx + \frac{x}{2}dy\right) = \int\int_R \left(\frac{1}{2} - \left[-\frac{1}{2}\right]\right)dA = \int\int_R dA = \text{area of } R$$

Example: Find the area of the region bounded by the curves $y = x^3$ and $y = x^{\frac{1}{2}}$



.

Let $C = C_1 + C_2$, where $C_2 : y = x^{\frac{1}{2}}\ x : 1 \to 0$ and $C_1 : y = x^3\ 0 \le x \le 1$. Then *C* is a closed curve which bounds the region. We shall use *x* as the parameter on *C* and the formula in the corollary. ⇒

$$\begin{aligned} A &= \frac{1}{2}\oint_C (x\,dy - y\,dx) \\ &= \frac{1}{2}\int_{C_1}[x(3x^2)dx - x^3dx] + \frac{1}{2}\int_{C_2}\left[x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)dx - x^{\frac{1}{2}}dx\right] \\ &= \frac{1}{2}\int_0^1 2x^3dx - \frac{1}{4}\int_1^0 x^{\frac{1}{2}}dx = \frac{5}{12} \end{aligned}$$

Example: Evaluate the line integral

$$\oint_C (x^3 + 2y)dx + (4x - 3y^2)dy$$

where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: $P = x^3 + 2y, \quad Q = 4x - 3y^2 \Rightarrow Q_x = 4, \quad P_y = 2$

By Green's Theorem:

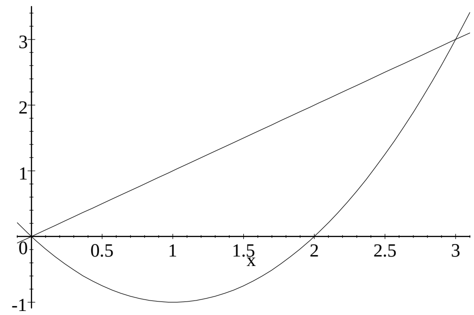
$$\oint_C Pdx + Qdy = \int\int_R (4-2)dA = 2\int\int dA$$

Therefore we need the area of the ellipse which is $\pi\, ab. \Rightarrow \oint = 2\pi\, ab.$

Example. Verify Green's theorem for

$$\oint_C 3xydx + 2x^2 dy$$

where C is the curve which bounds the region R above by $y = x$ and below by $y = x^2 - 2x$



Since $P = 3xy$ and $Q = 2x^2$, we see that $Q_x - P_y = 4x - 3x = x$. The curves intersect when $x = x^2 - 2x$ or $x^2 - 3x = x(x-3) = 0.$ Hence when $x = 0, 3.$ Thus

$$\int\int (Q_x - P_y)dA = \int_0^3 \int_{x^2-2x}^{x} xdydx = \frac{27}{4}$$

$C = C_1 + C_2$ where $C_1 : y = x^2 - 2x \quad C_2 : y = x \quad 0 \le x \le 3$

On C_1 :

$$\int_{C_1} Pdx + Qdy = \int_0^3 3x(x^2 - 2x)dx + \int_0^3 2x^2(2x-2)dx$$

On C_2 :

$$\int_{C_2} Pdx + Qdy = \int_3^0 3x(x)dx + \int_3^0 2x^2 dx$$

A straight forward calculation shows that the sum of these last two expressions also equals $\frac{27}{4}$.

Example. Use Green's theorem to evaluate

$$\int_C \left(2y + \sqrt{9 + x^3}\right)dx + (5x + e^{\arctan y})dy$$

where C is the circle $x^2 + y^2 = 4.$

Now $Q_x - P_y = 5 - 2 = 3. \Rightarrow$

$$\iint_{x^2+y^2\le 4} 3dA = 3(\text{area of a circle of radius 2}) = 3(4\pi) = 12\pi$$

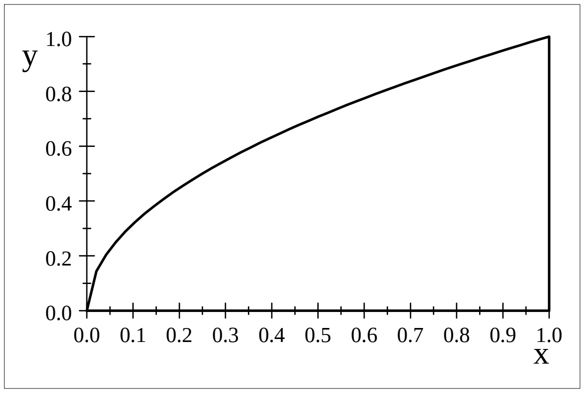
Example Evaluate

$$\oint_C (1+\tan x)dx + (x^2+e^y)dy$$

Where C is the positively oriented boundary of the region R enclosed by the curves $y = \sqrt{x}$, $x = 1$, and $y = 0$. Be sure to sketch C.

Solution:

The region enclosed by C is shown below.



We use Green's Theorem to evaluate the integral since C is a closed curve.

$$\oint_C (1+\tan x)dx + (x^2+e^y)dy = \int\int_R \left(\frac{\partial(x^2+e^y)}{\partial x} - \frac{\partial(1+\tan x)}{\partial y}\right)dA$$

$$= \int_0^1\int_0^{\sqrt{x}} (2x-0)dydx = 2\int_0^1 x^{\frac{3}{2}}dx = \frac{4}{5}$$