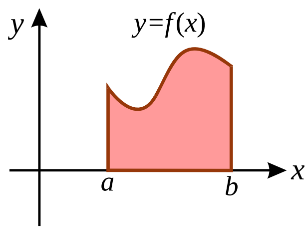
Ma 227 - MULTIPLE INTEGRATION

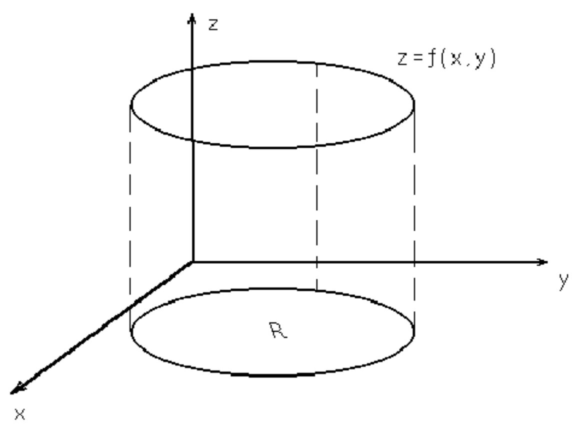
Remark: The concept of a function of one variable in which $y = g(x)$ may be extended to two or more variables. If z is uniquely determined by values of the variables x and y, then we say z is a function of x and y, and write $z = f(x,y)$. Thus for each pair of values x and y in the domain of f, $f(x,y)$ gives one value of z.

Double Integrals

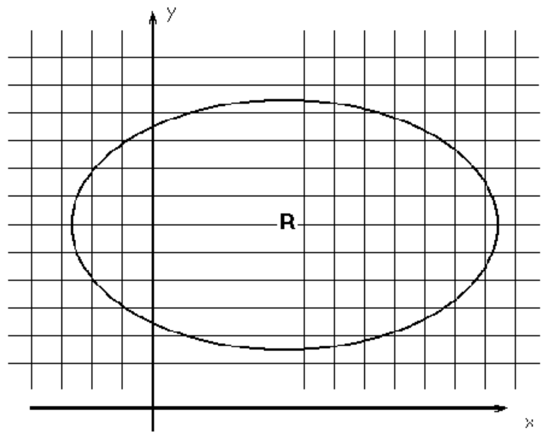
Recall that if $f(x) > 0$ then $\int_a^b f(x)dx$ represents the area under f between $x = a$ and $x = b$.



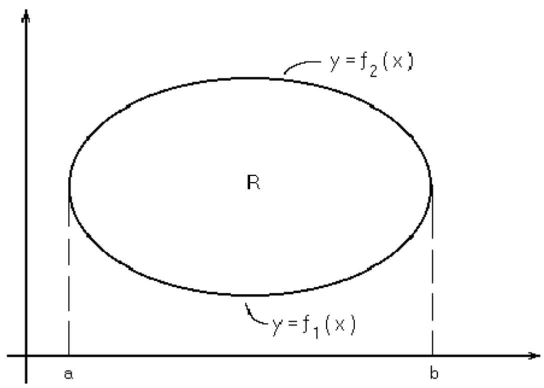
Now consider a function $f(x,y)$ of *two* variables x and y. Then $I = \int\int_R f(x,y)dA$ denotes the double integral over the region R of the function $f(x,y)$. Actually when f is positive I is the volume under f which is enclosed by f, its projection R onto the x,y –plane and the "shell of the projection".



If we imagine a grid in the x,y –plane then $\Delta A = \Delta x\Delta y = \Delta y\Delta x$ and $dA = dxdy = dydx \Rightarrow$
$I = \int\int_R f(x,y)dxdy = \int\int_R f(x,y)dydx.$



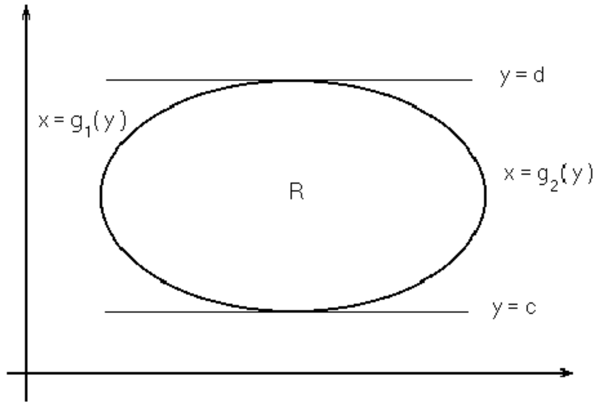
If we are given the boundaries of *R* in terms of *y* as a function of *x*, i.e.



Then

$$\iint_R f(x,y)dA = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)dy \right] dx.$$

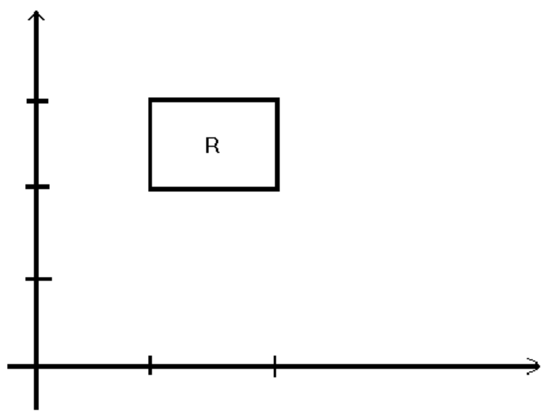
On the other hand if we are given the boundaries of *R* in a form in which *x* is a function of *y* as indicated below



Then

$$\iint_R f(x,y)dA = \int_c^d \left[\int_{g_1(y)}^{g_2(y)} f(x,y)dx \right] dy$$

Evaluate $\iint_R x^2y^3 dA$ where R is the region contained by the lines $x = 1, x = 2, y = 2,$ and $y = 3$.



$$\iint_R x^2y^3dA = \int_2^3\int_1^2 x^2y^3dxdy = \int_2^3 y^3\left[\int_1^2 x^2dx\right]dy = \int_2^3 y^3\left[\frac{x^3}{3}\right]_1^2 dy$$

$$= \int_2^3 y^3\left[\frac{8}{3} - \frac{1}{3}\right]dy = \frac{7}{3}\int_2^3 y^3dy = \frac{7}{3}\left[\frac{y^4}{4}\right]_2^3 = \frac{7}{12}[3^4 - 2^4] = \frac{455}{12}$$

We may also calculate the double integral by integrating with respect to y first

$$\iint_R x^2y^3dA = \int_1^2\int_2^3 x^2y^3dydx = \int_1^2 x^2\left[\int_2^3 y^3dy\right]dx = \int_1^2 x^2\left[\frac{y^4}{4}\right]_2^3 dx$$

$$= \frac{1}{4}\left[3^4 - 2^4\right]\int_1^2 x^2dx = \frac{65}{4}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{12}(65)[8-1] = \frac{455}{12}$$

Thus $\iint_R x^2y^3dydx = \iint_R x^2y^3dxdy$. This is true in general. However, one must make sure that the limits of integration are correct.

Evaluation of Double Integrals

Here are a couple of examples of how one evaluates more complicated double integrals.

Example

Evaluate

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy$$

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy = \int_1^{\ln 8}\left[\int_0^{\ln y} e^x dx\right]e^y dy = \int_1^{\ln 8}[e^x]_0^{\ln y} e^y dy = \int_1^{\ln 8} e^y[y-1]dy = \int_1^{\ln 8} ye^y dy - \int_1^{\ln 8} e^y dy$$

We use integration by parts to evaluate $\int_1^{\ln 8} ye^y dy$ with $u = y$ and $dv = e^y dy$

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy = [ye^y]_1^{\ln 8} - \int_1^{\ln 8} e^y dy - \int_1^{\ln 8} e^y dy = 8\ln 8 - e - 2e^y|_1^{\ln 8} = 8\ln 8 - e - 16 + 2e = 8\ln 8 + e - 16.$$

Example

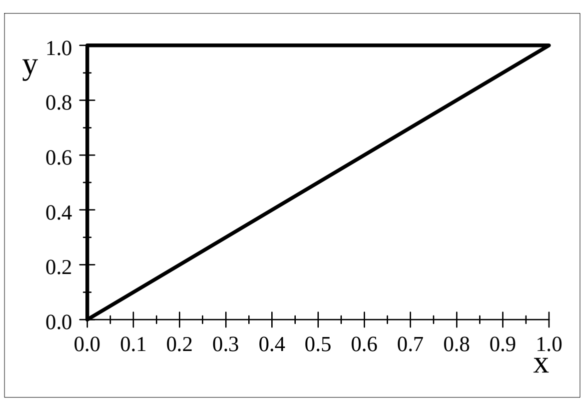
Evaluate

$$\iint_R x^2y^3dA$$

where R is the triangle with vertices at $(0,0),(0,1),(1,1)$.

Solution:

The triangle is shown below.



We will set up the integration in two ways. Consider first

$$\iint_R x^2 y^3 dxdy$$

Taking a horizontal strip parallel to the x –axis we see that x goes from the y –axis to the line $x = y$, whereas y goes from 0 to 1. Thus

$$\iint_R x^2 y^3 dxdy = \int_0^1 \int_0^y x^2 y^3 dxdy = \frac{1}{21}$$

If we now consider

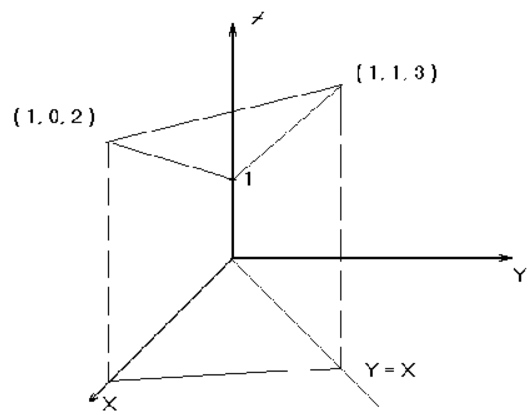
$$\iint_R x^2 y^3 dydx$$

then using a vertical strip parallel to the y –axis we see that y goes from the line $y = x$ to 1. so we have

$$\iint_R x^2 y^3 dydx = \int_0^1 \int_x^1 x^2 y^3 dydx = \frac{1}{21}$$

Example

Find the volume of the solid whose base is in the x,y –plane and is the triangle bounded by the x –axis, the line $y = x$ and the line $x = 1$, while the top of the solid is the plane $z = x + y + 1$.

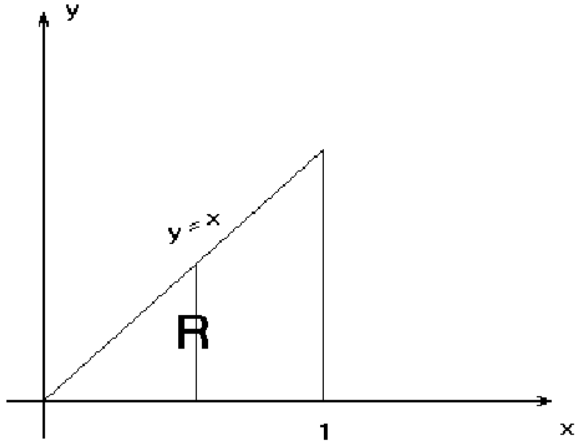


$$dV = f(x,y)dA = (x + y + 1)dxdy = (x + y + 1)dydx$$

Thus

$$V = \iint_R (x + y + 1)dA$$

where R is the base of the solid which is shown below.



The boundaries of R are $y = 0, y = x,$ and $x = 1$. Hence

$$V = \int_0^1 \int_0^x (x + y + 1)dydx = \int_0^1 \left[xy + \frac{y^2}{2} + y \right]_0^x dx = \int_0^1 \left[x^2 + \frac{x^2}{2} + x \right] dx$$

$$= \int_0^1 \left(\frac{3}{2}x^2 + x \right) dx = \left[\frac{3}{2}\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1.$$

Note that another expression for the volume is

$$V = \int_0^1 \int_y^1 (x + y + 1)dxdy.$$

Properties of Double Integrals

Double integrals have the same properties as integrals of one variable. For example, if c_1 and c_2 are constants, then

$$\iint_R [c_1 f(x,y) + c_2 g(x,y)]dA = c_1 \iint_R f(x,y)dA + c_2 \iint_R g(x,y)dA.$$

(1) $\int_{-1}^5 \int_{x-1}^x (4xe^y - 3y\sin x)dydx =$
$-\frac{\frac{1}{2}\left(-32 + 32e^{-1} + 6(\sin 5)e^{-5} - 27(\cos 5)e^{-5} - 16e^{-6} + 16e^{-7} + 6(\sin 1)e^{-5} - 9(\cos 1)e^{-5}\right)}{e^5}$

whereas

(2) $4\int_{-1}^5 \int_{x-1}^x xe^y dydx = -8\left(-2 + 2e^{-1} - e^{-6} + e^{-7}\right)e^5$

and

(3) $-3\int_{-1}^5 \int_{x-1}^x y\sin x dydx = -3\sin 5 + \frac{27}{2}\cos 5 - 3\sin 1 + \frac{9}{2}\cos 1$

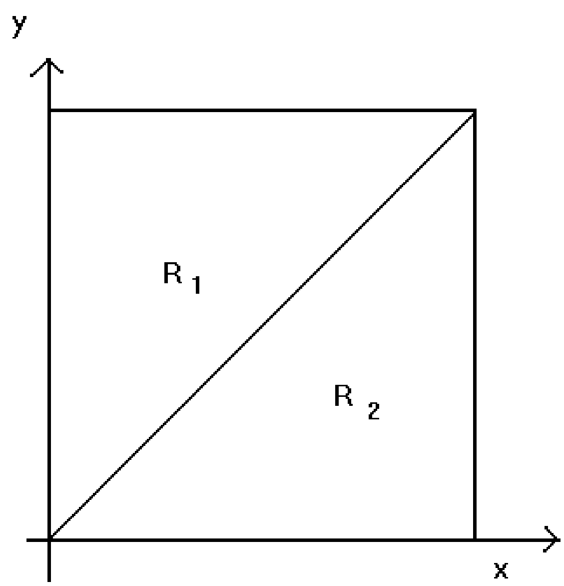
Adding the results given by (2) and (3) gives (1) after a bit of algebra.

If R is a closed region which can be decomposed into regions R_1 and R_2 and f is continuous over R, then

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA.$$

Example

Let R be the rectangular region $0 \le x,y \le 1$ shown below consisting of the triangles R_1 and R_2.



We shall show that

$$\iint_R x^2y^3dA = \iint_{R_1} x^2y^3dA + \iint_{R_2} x^2y^3dA$$

Now

$$\iint_R x^2y^3dA = \int_0^1\int_0^1 x^2y^3dxdy = \frac{1}{12}$$

or

$$\iint_R x^2y^3dA = \int_0^1\int_0^1 x^2y^3dydx = \frac{1}{12}$$

The triangle R_1 is given by $0 \le x \le y, 0 \le y \le 1$ so

$$\iint_{R_1} x^2y^3dA = \int_0^1\int_0^y x^2y^3dxdy = \frac{1}{21}$$

or

$$\iint_{R_1} x^2y^3 dA = \int_0^1 \int_x^1 x^2y^3 dydx = \frac{1}{21}$$

Triangle R_2 is given by $0 \le y \le x, 0 \le x \le 1$ so

$$\iint_{R_2} x^2y^3 dA = \int_0^1 \int_y^1 x^2y^3 dxdy = \frac{1}{28}$$

or

$$\iint_{R_2} x^2y^3 dA = \int_0^1 \int_0^x x^2y^3 dydx = \frac{1}{28}$$

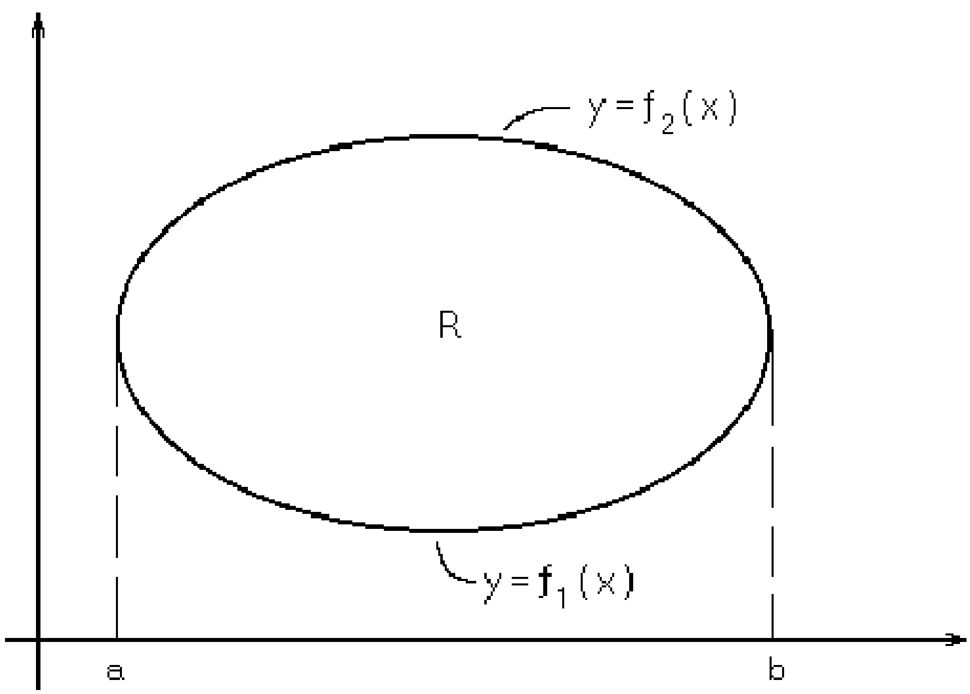
Finally $\frac{1}{21} + \frac{1}{28} = \frac{1}{12}$, which is the result we got before.

Special Case-Area by Integration

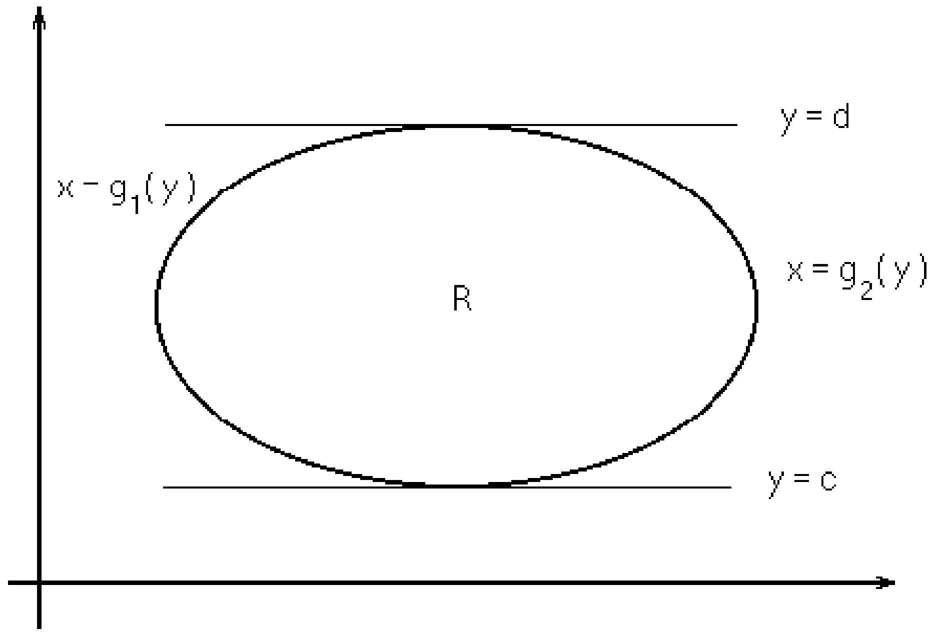
The special case of $\int\int_R f(x,y)dx\, dx$ when $f = 1$ is

$$\int\int_R dA = \int\int_R dxdy = \int\int_R dydx.$$

In this case the double integral represents the area of the region R in the x,y –plane.



$$Area = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx$$



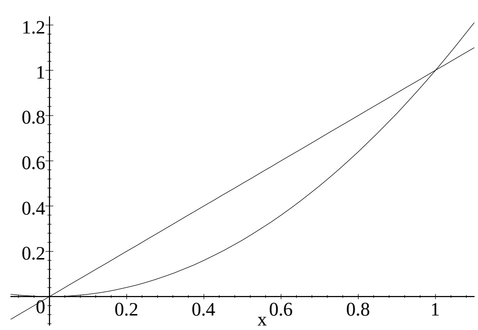
$$\text{Area} = \int_c^d \int_{g_1(y)}^{g_2(y)} dxdy$$

Example

The integral $\int_0^1 \int_{x^2}^{x} dy\, dx$ represents the area of a region of the x, y – plane. Sketch the region and express the same area as a double integral with the order of integration reversed.

Solution:

The inner integral varies from $y = x^2$ to $y = x$. Integral gives area of vertical strip between x and $x + dx$ for values of x from 0 to 1.



Change order and take integration first. Then x goes from y to $\sqrt{y}$ to give a horizontal strip between y and $y + dy$. Thus

$$\int_0^1 \int_{x^2}^x dydx = \int_0^1 \int_y^{\sqrt{y}} dxdy.$$

$$\int_0^1 \int_y^{\sqrt{y}} dxdy = \int_0^1 (\sqrt{y} - y)dy = \frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^2}{2}|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

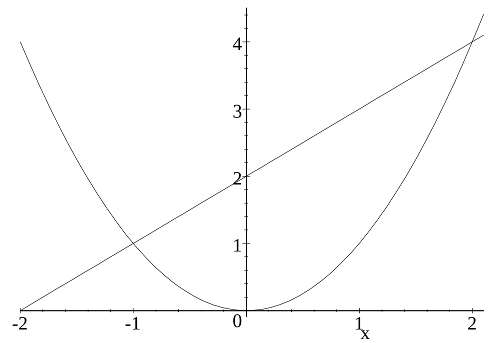
Also

$$\int_0^1 \int_{x^2}^x dydx = \int_0^1 (x - x^2)dx = \frac{1}{6}$$

which checks.

Example

Find the area bounded by the parabola $y = x^2$ and the line $y = x + 2$.



Solution:

The parabola and line intersect where $y = x^2 = x + 2$

$\Rightarrow\ x^2 - x - 2 = 0$ or $(x-2)(x+1) = 0$ Thus $x = 2,\quad x = -1$ are the x coordinates of the points of intersection. $x = 2 \Rightarrow y = 4;$ whereas$x = -1 \Rightarrow y = 1$

We shall first find the area as

$$\iint_R dxdy.$$

When y is between 0 and 1, x goes from $-\sqrt{y}$ to $\sqrt{y} \Rightarrow \int_0^1 \int_{-\sqrt{y}}^{+\sqrt{y}} dx\, dy$

When y is between 1 and 4, x goes from $y - 2$ to $\sqrt{y} \Rightarrow \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$. Thus

$$A = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy.$$

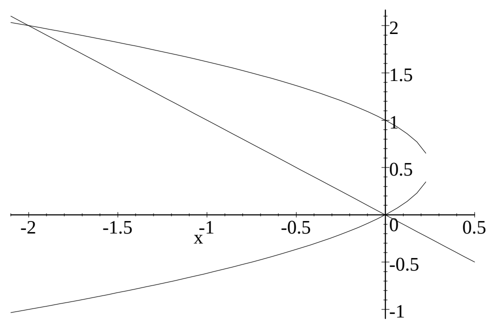
We now set up the expression for the area the other way. $\iint_R dy\, dx =$

$\int_{-1}^2 \int_{x^2}^{x+2} dy\, dx = \int_{-1}^2 y|_{x^2}^{x+2} dx = \int_{-1}^2 (x + 2 - x^2)dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}|_{-1}^2$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 9 - \frac{9}{3} - \frac{1}{2} = 5\frac{1}{2}.$$

Example

Find the area between the parabola $x = y - y^2$ and the line $x + y = 0$, that is the line $y = -x$.



Solution:

Now $x = y - y^2$ or $x = -(y^2 - y)$. Completing the square) $\Rightarrow x = -(y^2 - y + \frac{1}{4}) + \frac{1}{4}$ or $x - \frac{1}{4} = -(y - \frac{1}{2})^2$. Hence the parabola passes through $(\frac{1}{4}, \frac{1}{2})$. Now $x = 0 \Rightarrow y(1 - y) = 0 \Rightarrow y = 0$ and $y = 1$. Thus the parabola goes through the points $(0,0)$, $(0,1)$, and $(\frac{1}{4}, \frac{1}{2})$. We now find the points where the line and the parabola intersect. We have $x = y - y^2$ and $x = -y \Rightarrow -y = y - y^2$ or $0 = 2y - y^2 = y(2 - y). \Rightarrow y = 0$ or $y = 2$. The points of intersection are therefore $(0,0)$ and $(-2,2)$. We again set up the expression for area in two ways. First consider

$$\iint_R dxdy.$$

$$A = \int_0^2 \int_{-y}^{y-y^2} dxdy = \int_0^2 (y - y^2 + y)dxdy = \int_0^2 (2y - y^2)dy = y^2 - \frac{y^3}{3}\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$$

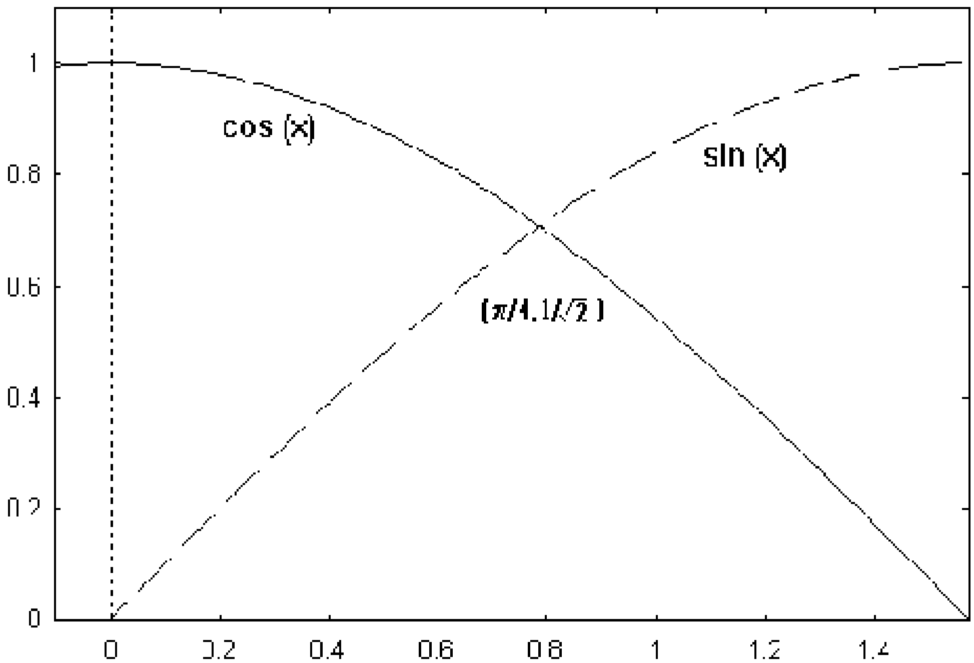
Now for

$$\iint_R dydx.$$

$y - \frac{1}{2} = \pm\sqrt{\frac{1}{4} - x}$ on the parabola. Thus

$$A = \int_{-2}^{0} \int_{-x}^{\sqrt{\frac{1}{4}-x}+\frac{1}{2}} dydx + \int_0^{\frac{1}{4}} \int_{-\sqrt{\frac{1}{4}-x}+\frac{1}{2}}^{\sqrt{\frac{1}{4}-x}+\frac{1}{2}} dydx$$

Change the order of integration in $\int_0^{\frac{\pi}{4}} dx \int_{\sin x}^{\cos x} f(x,y)dy$



$\int_0^{\frac{1}{\sqrt{2}}} dy \int_0^{\sin^{-1}y} f(x,y)dx + \int_{\frac{1}{\sqrt{2}}}^{x} dy \int_0^{\cos^{-1}y} f(x,y)dx.$

Polar coordinates-change of variables

Recall that given a point (x,y) we may assign to this point new coordinates (r,θ) as follows:

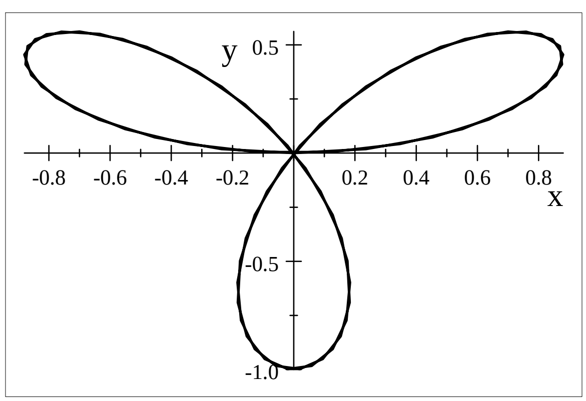
$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \qquad r = \sqrt{x^2+y^2}$$

If $r > \theta$ and θ are given, then they uniquely determine a point in the x,y –plane. An equation of the form $r = f(\theta)$ determines a curve in the (x,y) –plane. This topic was discussed in Ma 116.

Example

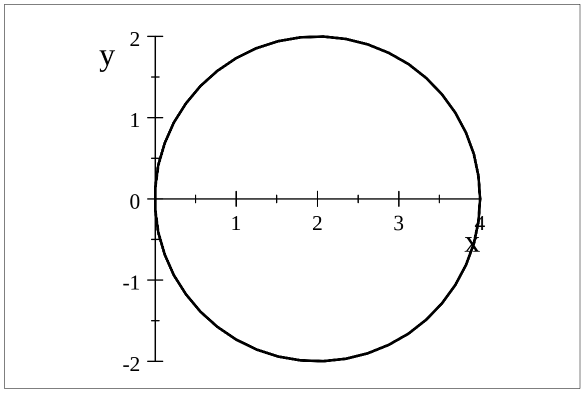
Graph $r = \sin 3\theta$



$r = \sin 3\theta$

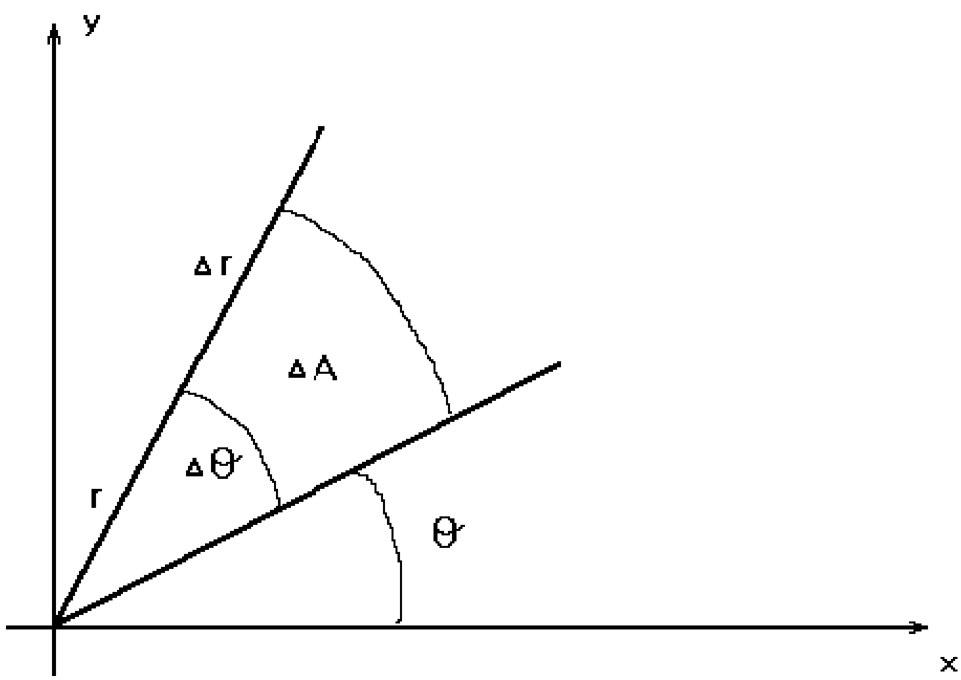
Example

Graph $r = 4\cos\theta$



$\Rightarrow\ r^2 = 4r\cos\theta = 4x$. Since $r^2 = x^2 + y^2$ so that $x^2 + y^2 = 4x$ or $x^2 - 4x + y^2 = 0$, which is the circle $(x-2)^2 + y^2 = 4$ centered at $(2,0)$ with radius 2.

Area Using Polar Coordinates



Recall $dA = dx\,dy$. Now from the figure

$$\begin{aligned}\Delta A &= \frac{1}{2}(r + \overline{\Delta r})^2\Delta\theta - \frac{1}{2}r^2\Delta\theta \\ &= \frac{1}{2}(r^2 + 2\overline{\Delta r}r + \Delta r^2)\Delta\theta - \frac{1}{2}r^2\Delta\theta \\ &= \overline{\Delta r}r\Delta\theta + \frac{1}{2}\Delta r^2\Delta\theta \approx r\overline{\Delta r}\Delta\theta\end{aligned}$$

$\Rightarrow$

$$dA = rdrd\theta$$

$\Rightarrow$

$$\iint_R f(x,y)dxdy \;\Rightarrow\; \iint_R F(r,\theta)rdrd\theta.$$

Example

Find the area of the circle $(x-2)^2 + y^2 = 4$.

Solution:

We know that $A = \pi r^2 = 4\pi$. Using double integration in polar coordinates, we have

$$A = 2\int_0^{\frac{\pi}{2}}\int_0^{4\cos\theta} rdrd\theta = 2\int_0^{\frac{\pi}{2}} \frac{r^2}{2}\Big|_0^{4\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} [16\cos^2\theta]d\theta$$
$$= 16\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 8\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 8\left[\frac{\pi}{2}\right] = 4\pi.$$

Example

Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$.

Switching to polar coordinates, we have

$$\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}rdrd\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta = \int_0^{\frac{\pi}{2}} \frac{-e^{r^2}}{2}\Big|_0^\infty d\theta = +\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{4}$$

Example

(i) Find the equations in polar coordinates of the curves $x^2 + y^2 = 2y$ and $x^2 + y^2 = 2x$ and graph the curves.

Solution:

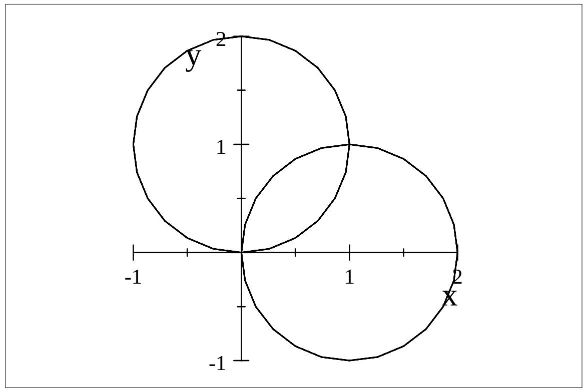
The two curves are given by

$$r = 2\sin\theta$$

and

$$r = 2\cos\theta$$

The graphs are given below.



(ii) Give an integral or integrals in polar coordinates for the area between the two curves.

Solution:

$$A = \iint_R r dr d\theta$$

where R is the region common to both circles. The two circles intersect when

$$2\cos\theta = 2\sin\theta$$

or when

$$\tan\theta = 1$$

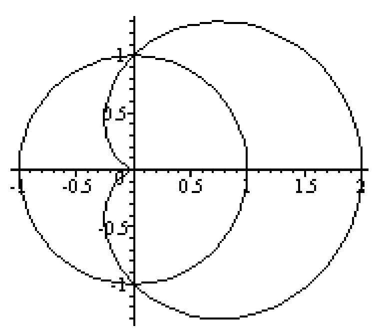
That is, at $\theta = \frac{\pi}{4}$. r goes from 0 to the circle $r = 2\sin\theta$, for $0 \le \theta \le \frac{\pi}{4}$ and from 0 to $r = 2\cos\theta$ for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$. Thus we need two integrals to express the area.

$$A = \int_0^{\frac{\pi}{4}} \int_0^{2\sin\theta} r dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r dr d\theta$$

$$= \frac{1}{2}\pi - 1$$

Example

Find the area which lies inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$. Use double integration. The figure below shows the two curves with $a = 1$.



$$A = \int\int r dr d\theta = 2\int_0^{\frac{\pi}{2}} \int_a^{a(1+\cos\theta)} r dr d\theta = 2a^2 + \frac{1}{4}a^2\pi$$

Example

Give an integral in polar coordinates which represents the area of the region R that lies outside the circle $r = a$ and inside the circle $r = 2a\sin\theta$.

Solution:

We must sketch R.

First, $x = r\cos\theta,\ y = r\sin\theta.$ Thus the circle $r = a$ is centered at the origin and has radius a.

We rewrite the equation of the other circle.

$$r^2 = 2ar\sin\theta$$

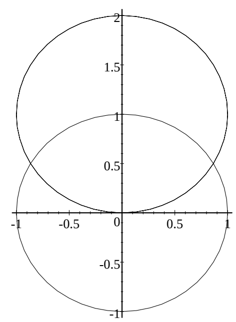
Thus

$$x^2 + y^2 = 2ay$$

or

$$x^2 + (y - a)^2 = a^2$$

This circle passes through the origin, is centered on the y-axis at $(0, a)$ and has radius a. For convenience, $a = 1$ in the picture below.



To find the limits of integration, we have to equate the expressions for the two circles.

$a = 2a\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$, where the circles intercept. So θ lies between these two values. On the other hand, r goes from a to $2a\sin\theta$ for these values of θ.

The integral for the area is:

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{a}^{2a\sin\theta} r\,dr\,d\theta = \frac{1}{3}a^2\pi + \frac{1}{2}a^2\sqrt{3}$$

Triple Integrals

We shall now discuss a logical extension of the double integral. Consider

$$\iiint_V F(x,y,z)dV = \iiint_V F(x,y,z)dxdydz = \iiint_V F(x,y,z)dydxdz = \iiint_V F(x,y,z)dzdxdy = \cdots \text{ etc.}$$

This is clearly merely an extension of the double integral.

Example

Evaluate

$$I = \int_0^1 \int_0^{1-x} \int_0^{2-x} xyz dz dy dx$$

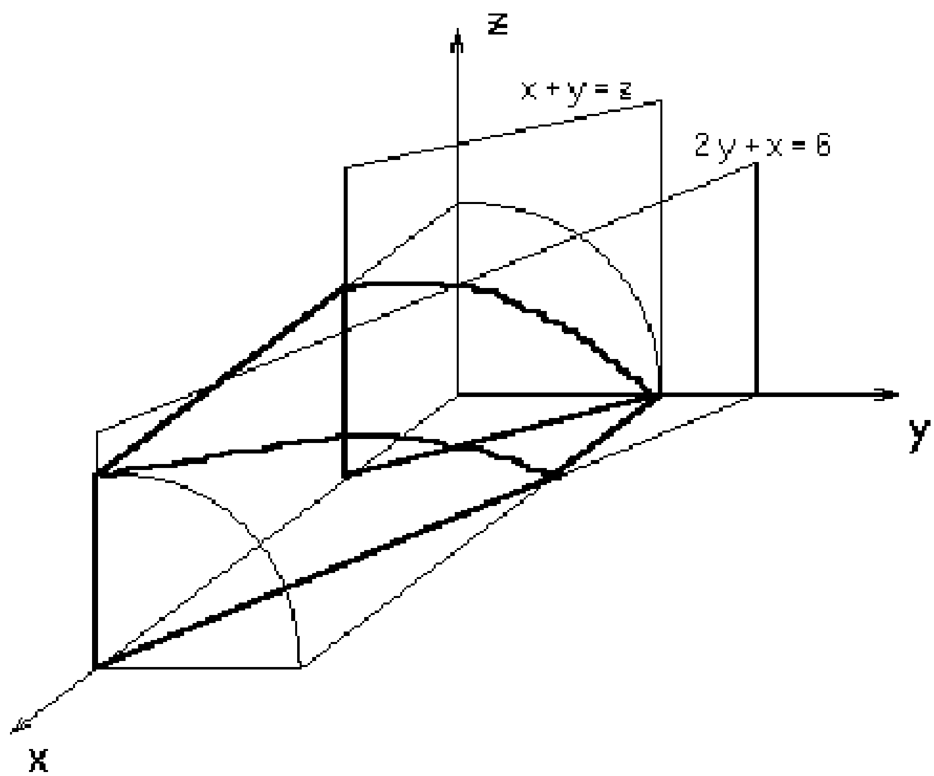
$$I = \int_0^1 \left[\int_0^{1-x} \left\{ \int_0^{2-x} xyz\, dz \right\} dy \right] dx = \int_0^1 \left[\int_0^{1-x} xy \frac{(2-x)^2}{2} dy \right] dx$$

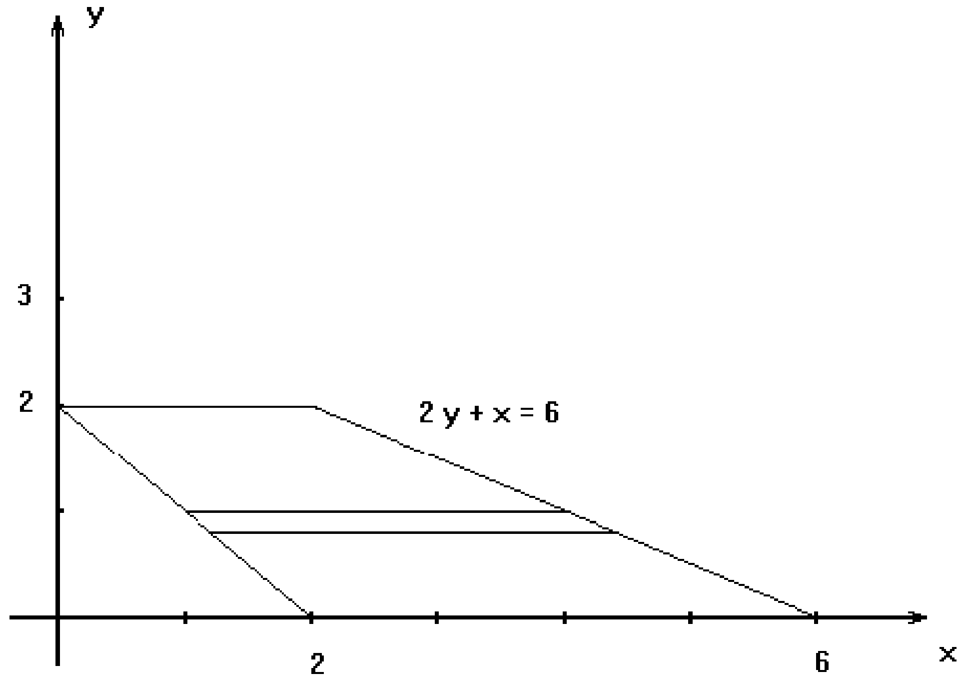
$$= \int_0^1 \frac{xy^2(2-x)^2}{4} \Big|_{y=0}^{y=1-x} dx = \int_0^1 \left[\frac{1}{4} x(x-1)^2 (x-2)^2 \right] dx$$

$$= \frac{1}{4} \int_0^1 (4x - 12x^2 + 13x^3 - 6x^4 + x^5) dx = \frac{13}{240}$$

Example

Compute the triple integral of $F(x, y, z) = z$ over the region in the first octant bounded by the planes $y = 0$, $z = 0$, $x + y = 2$, $2y + x = 6$ and the cylinder $y^2 + z^2 = 4$.





$$I = \int\int\int z dv = \int_0^2 \int_{2-y}^{6-2y} \int_0^{\sqrt{4-y^2}} z dz dx dy$$

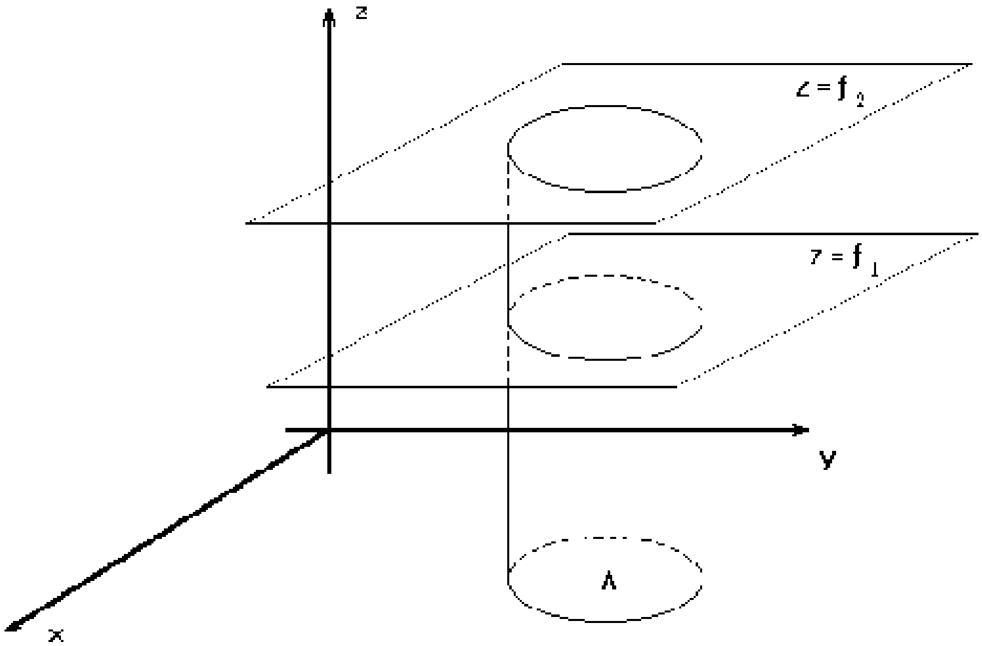
: $\frac{26}{3}$

Volume

The case $f = 1$, i.e.

$$\int\int\int dV$$

is of particular interest. It yields the volume between two surfaces. To see this suppose a region of x, y, z –space is bounded below by the surface $z = f_1(x, y)$, above by the surface $z = f_2(x, y)$ and laterally by a cylinder C with elements parallel to the z axis. Let A denote the region of the x, y –plane enclosed by the cylinder C.



Then the volume of the region is

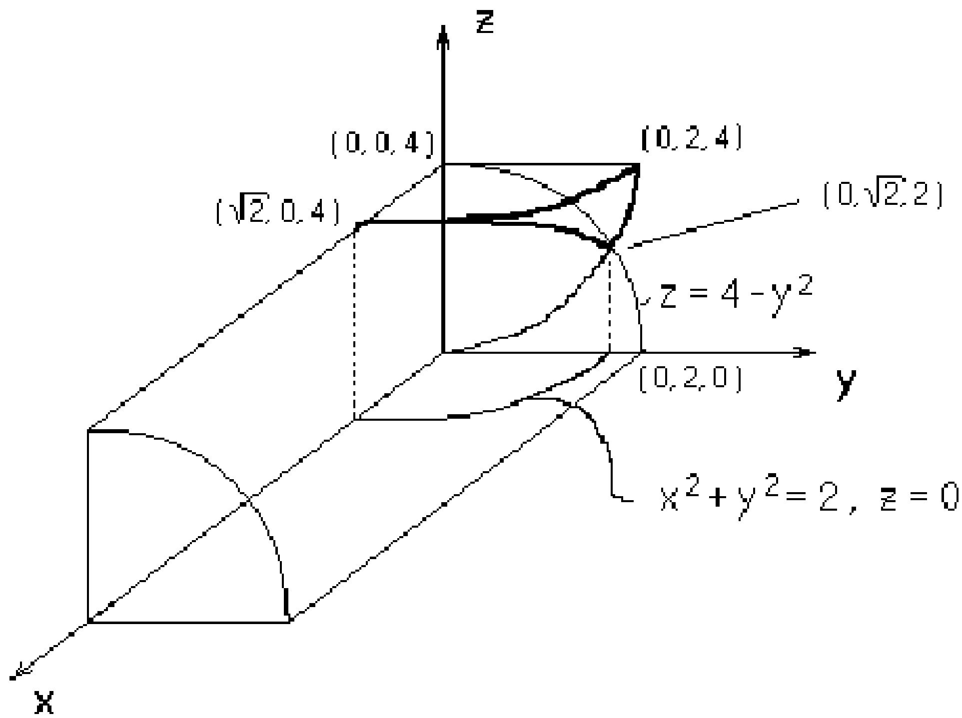
$$V = \iint_A \int_{f_1(x,y)}^{f_2(x,y)} dzdydx.$$

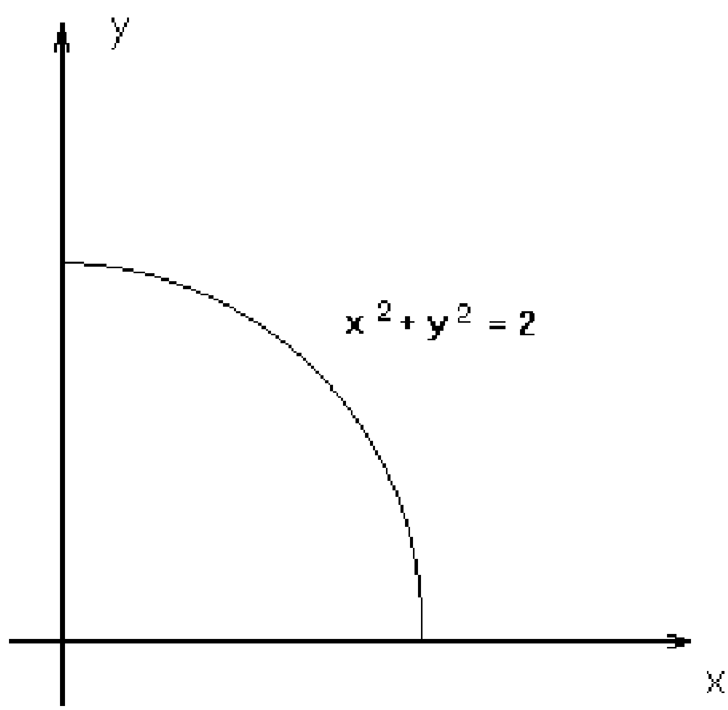
The x and y limits of integration extend over the region A. To get the x, y limits it is usually desirable to draw the x, y –plane view of the solid. Often one can get the boundary of A by eliminating z from $z = f_1(x,y)$ and $z = f_2(x,y)$, i.e. from $f_1(x,y) = f_2(x,y)$. In the x, y –plane this represents the boundary of A.

Example

Find the volume bounded by the paraboloid $z = 2x^2 + y^2$ and the parabolic cylinder $z = 4 - y^2$.
$2x^2 + y^2$

Solution:





z: From paraboloid to cylinder $\Rightarrow$ $2x^2 + y^2 \rightarrow 4 - y^2$

y: From 0 to $\sqrt{2 - x^2}$; gotten by eliminating z from 2 equations

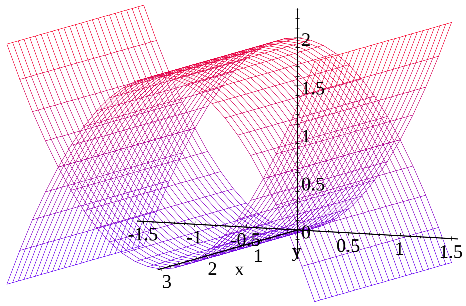
x: From 0 to $\sqrt{2}$ set; $y = 0$ in $x^2 + y^2 = 2$
$\Rightarrow$

$$V = 4\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} dzdydx = 4\pi$$

Example

Find the volume of the solid region D between the parabolic cylinders $z = y^2$ and $z = 2 - y^2$ for $0 \le x \le 3$. Sketch D.

Solution:



$2 - y^2$;y^2

We obtain the intersection lines of the two surfaces: $z_1 = z_2 = 2 - y^2 = y^2 \Rightarrow y = \pm 1$.

The limits of integration are then: $0 \le x \le 3$; $-1 \le y \le 1$;

Then

$$\begin{aligned} V &= \int_0^3 \int_{-1}^1 \int_{y^2}^{2-y^2} dzdydx \\ &= \int_0^3 \int_{-1}^1 \left[\left(2 - y^2\right) - y^2 \right] dydx \\ &= \int_0^3 \int_{-1}^1 \left(2 - 2y^2\right) dydx \\ &= 8 \end{aligned}$$

Cylindrical and Spherical Coordinates

Cylindrical coordinates are related to Cartesian coordinates via

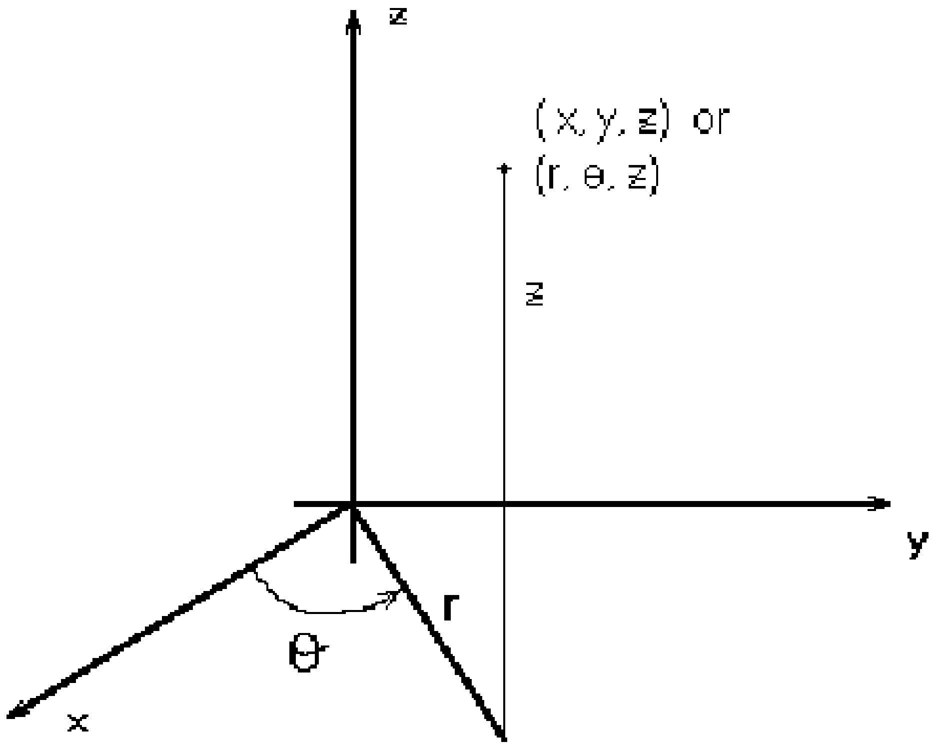
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

The relationship between a volume element in the two systems is

$$dV = dxdydz \to rdrd\theta dz,$$

that is

$$\int\int\int dV = \int\int\int r dr d\theta dz$$



Spherical coordinates are related to Cartesian coordinates via $x, y, z \rightarrow \rho, \theta, \phi$ where

$$x = \rho\cos\theta\sin\phi \qquad y = \rho\sin\theta\sin\phi \qquad z = \rho\cos\phi \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$$

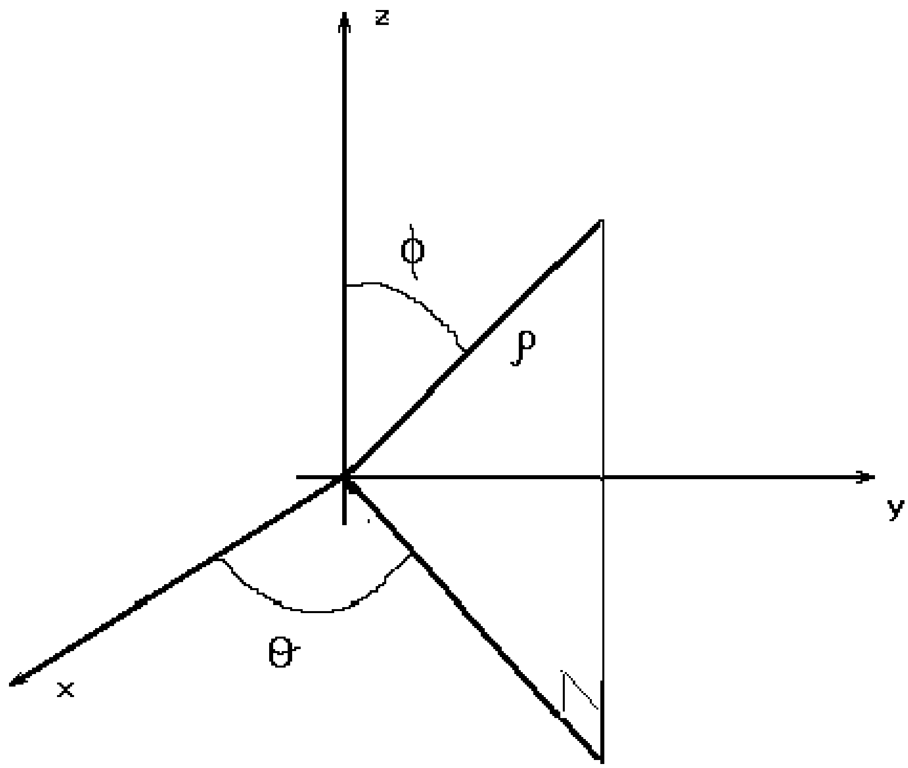
The relationship between a volume element in the two systems is

$$dV = dxdydz \rightarrow \rho^2 \sin\phi d\rho d\theta d\theta,$$

that is

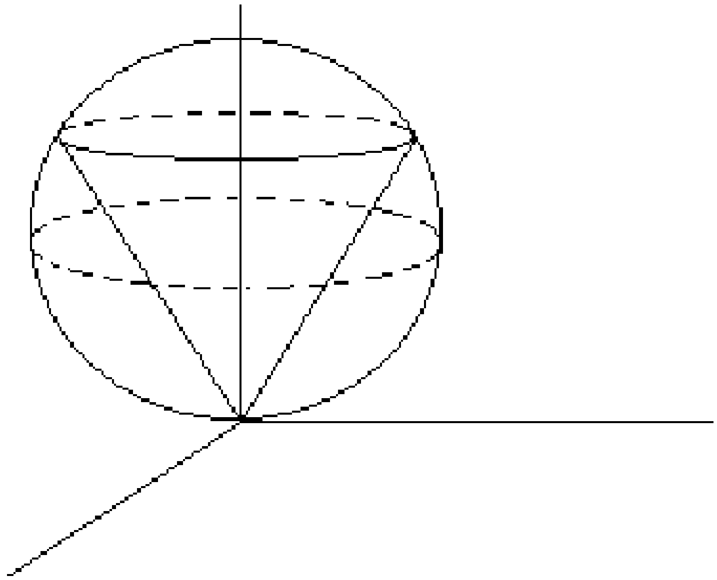
$$\int\int\int dV = \int\int\int \rho^2 \sin\phi d\rho d\theta d\theta.$$

It is important to keep in mind that ϕ is measured from the z axis and thus varies only from 0 to π.



Example

Find the volume above the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 2az$.



We shall use spherical coordinates.

Cone: $z^2 = x^2 + y^2$

$z = \rho\cos\phi \qquad x = \rho\cos\theta\sin\phi \qquad y = \rho\sin\theta\sin\phi$

The equation of the cone $\Rightarrow$ $\rho^2\cos^2\phi = \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$ or $\cos^2\phi = \sin^2\phi$

$$\Rightarrow \tan\phi = 1 \Rightarrow \phi = \pm 45° = \frac{\pi}{4} \text{ or } \phi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}.$$

Sphere: $x^2 + y^2 + z^2 - 2az = 0$ or $x^2 + y^2 + (z-a)^2 = a^2$. Center at $(0, 0, a)$.

$\Rightarrow$

$$\rho^2 - 2a\rho\cos\phi = 0 \text{ or } \rho = 2a\cos\phi.$$

We see that ϕ goes from 0 to $\frac{\pi}{4}$, θ from 0 to 2π. and ρ from 0 to $\rho = 2a\cos\phi$.

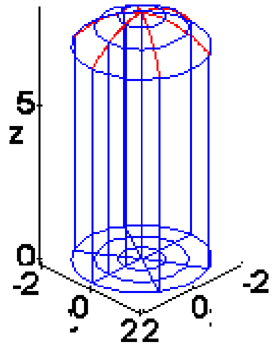
Hence

$$Volume = \iiint \rho^2\sin\phi dV_{\rho\theta\phi} = \int_0^{\frac{\pi}{4}} \int_0^{2a\cos\phi} \int_0^{2\pi} \rho^2\sin\phi d\theta d\rho d\phi = \pi a^3$$

Example

Give the expression in *cylindrical* coordinates for the volume of the solid inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$. Sketch the volume. Do *not* evaluate this expression.

SOLUTION



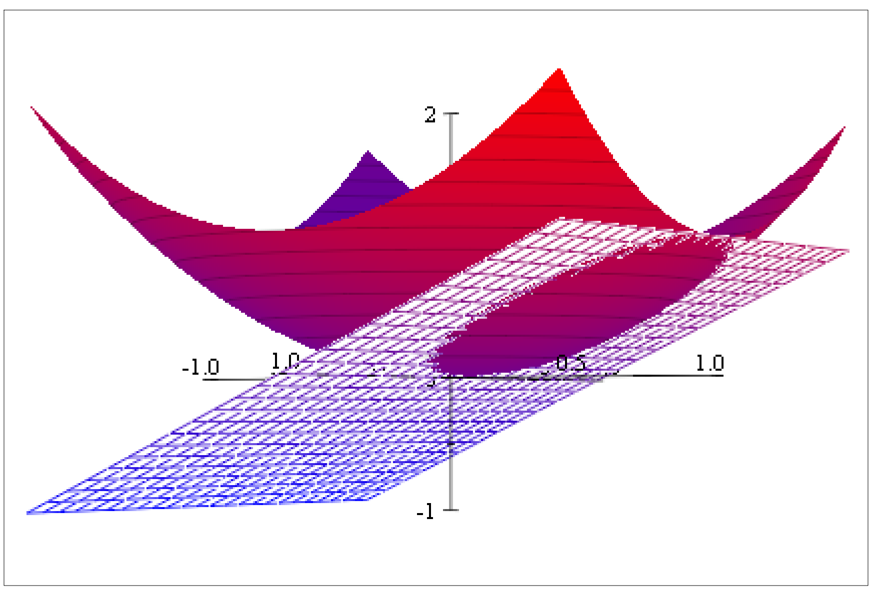
The ellipsoid intersects the x, y-plane in the circle $x^2 + y^2 = 16$. Thus, our region is bounded by the circle $x^2 + y^2 = 4$. So , in polar coordinates we have the equation $r = 2$. Next, we can solve the equation of the ellipsoid $4x^2 + 4y^2 + z^2 = 64$ for z, i.e., $z = \pm 2\sqrt{-x^2 - y^2 + 16}$ which can be rewritten in polar coordinates as $z = \pm 2\sqrt{16 - r^2}$. The volume of the solid can now be written as:

$$2\int_0^{2\pi}\int_0^2\int_0^{+2\sqrt{16-r^2}} r\,dz\,dr\,d\theta$$

Additional Cylindrical and Spherical Coordinates Examples

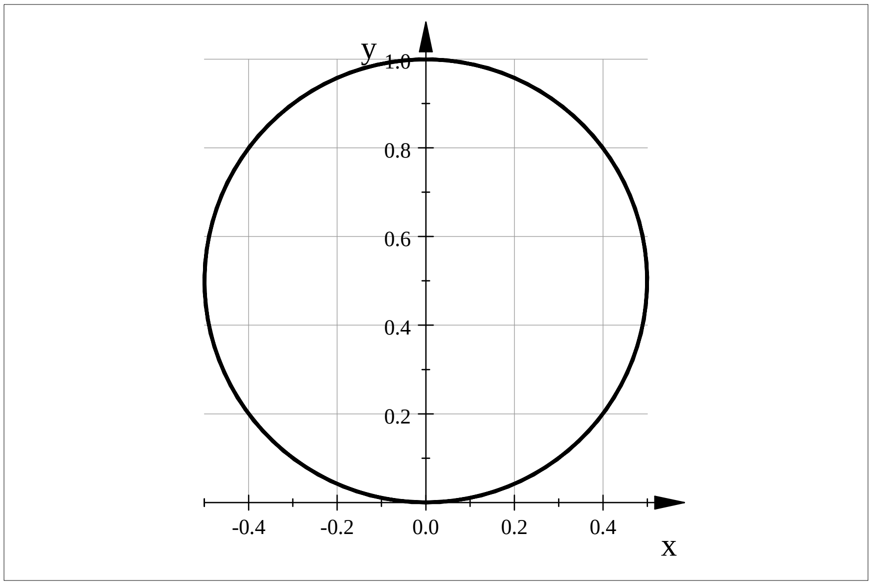
Example Give an expression in cylindrical coordinates for the volume of the solid T bounded above by the plane $z = y$ and below by the paraboloid $z = x^2 + y^2$. Sketch T. Do *not* evaluate this integral.

y



Solution: In cylindrical coordinates the plane has the equation $z = r\sin\theta$ and the paraboloid has the equation $z = r^2$. The two surfaces intersect when $y = x^2 + y^2$, that is the circle $x^2 + y^2 - y = 0$ or $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. This circle is only in first and second quadrants. The equation of this circle is

$r = \sin\theta$ in polar coordinates.



$$\text{Volume} = \int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta} r\,dz\,dr\,d\theta$$

Example Evaluate

$$\iiint_V \cos\left(x^2+y^2+z^2\right)^{\frac{3}{2}} dV$$

where V is the unit ball.

Solution: V is given by $x^2+y^2+z^2 \leq 1.$ In spherical coordinates the equation for V is $\rho = 1.$ Thus

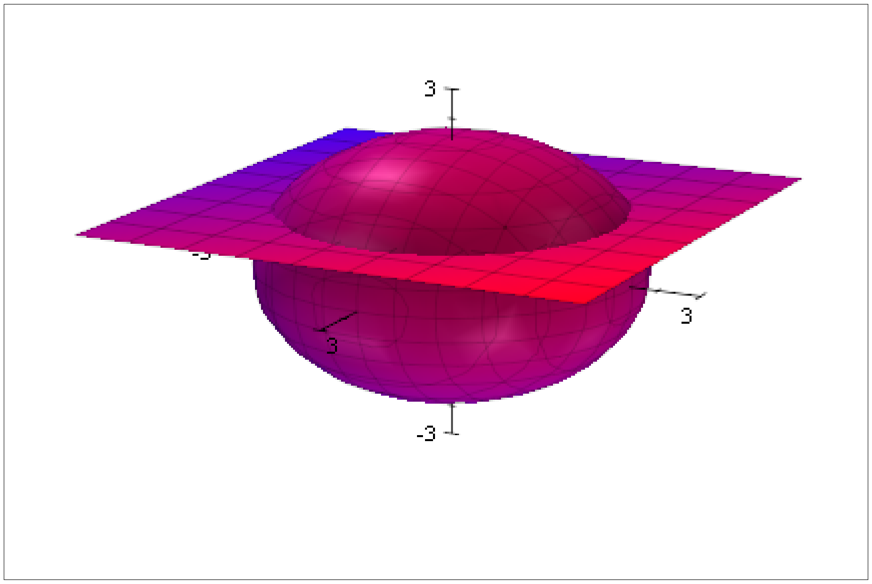
$$\iiint_V \left[\cos\left(x^2+y^2+z^2\right)\right]^{\frac{3}{2}} dV_{xyz} = \iiint_V \cos\left(\rho^2\right)^{\frac{3}{2}}\rho^2\sin\phi dV_{\rho\theta\phi}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \cos\left(\rho^3\right)\rho^2\sin\phi d\rho d\phi d\theta = \frac{4}{3}\pi\sin 1$$

Example 1.) Set up, but DO NOT INTEGRATE, a triple integral to find the volume of the solid bounded above by $x^2+y^2+z^2 = 5$ and below by $z = 1$ using spherical coordinates.

Solution: The region of integration is shown below. One uses Plot 3D, Implicit to get the picture.

$x^2+y^2+z^2 = 5$



ρ will go from the plane $z = 1$ to the sphere $x^2 + y^2 + z^2 = 5$.

In spherical, $x^2 + y^2 + z^2 = 5 \Rightarrow \rho = \sqrt{5}$

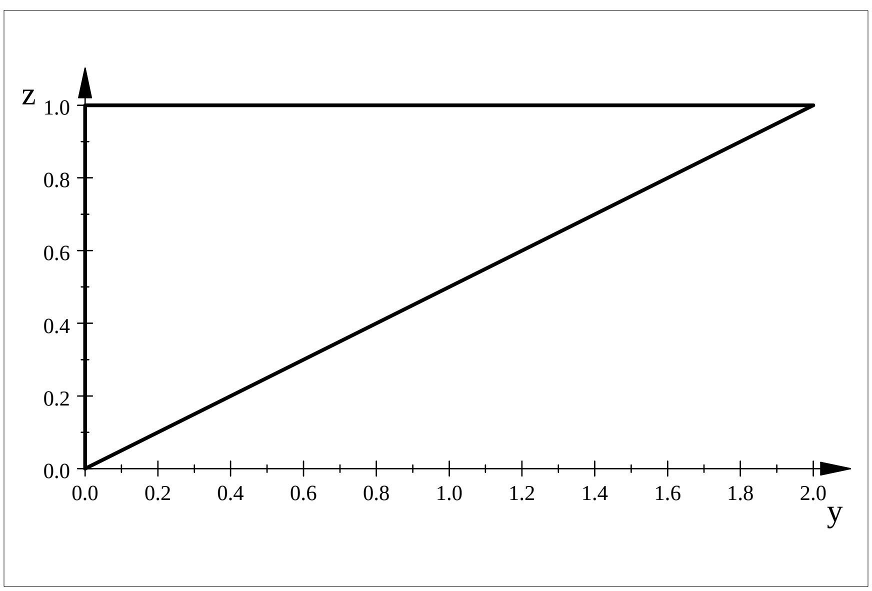
Also, $z = 1 \Rightarrow \rho\cos\phi = 1 \Rightarrow \rho = \sec\phi$.

So, $\sec\phi \leq \rho \leq \sqrt{5}$.

For ϕ, we can form a right triangle with hypotenuse $\sqrt{5}$ (the radius of the sphere) and vertical side 1 which is the distance from the origin to $z = 1$. So the horizontal side is 2.

$\sqrt{5} = 2.2361$

x



Therefore, $\tan\phi = 2 \Rightarrow \phi = \arctan 2$.

So, $0 \leq \phi \leq \arctan 2$.

The volume is:

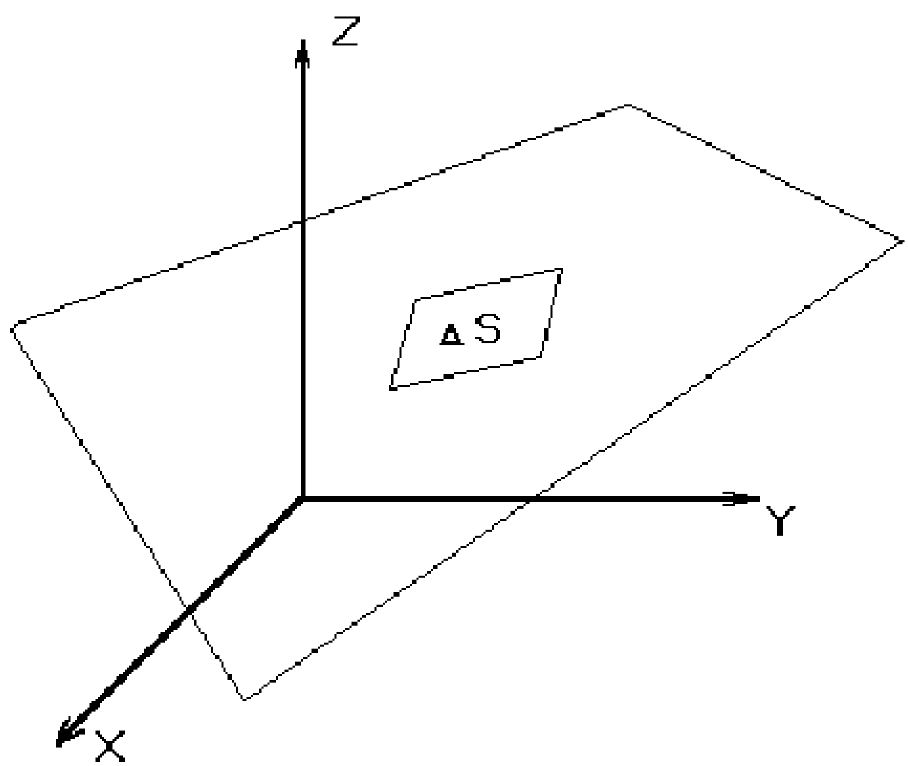
$$V = \int_0^{2\pi} \int_0^{\arctan 2} \int_{\sec\phi}^{\sqrt{5}} p^2 \sin\phi d\rho d\phi d\theta$$

Surface Integrals

It is often necessary to integrate a function over a curved surface. Such integrals are called surface integrals.
Let $z = \vartheta(x,y)$ describe a particular surface S. Let $f = f(x,y,z)$ be a given function. We desire to integrate f over S, i.e. to evaluate

$$\int\int_S f(x,y,z)dS = \int\int_S f(x,y,\vartheta(x,y))dS.$$

Here dS comes from dividing S into pieces ΔS.

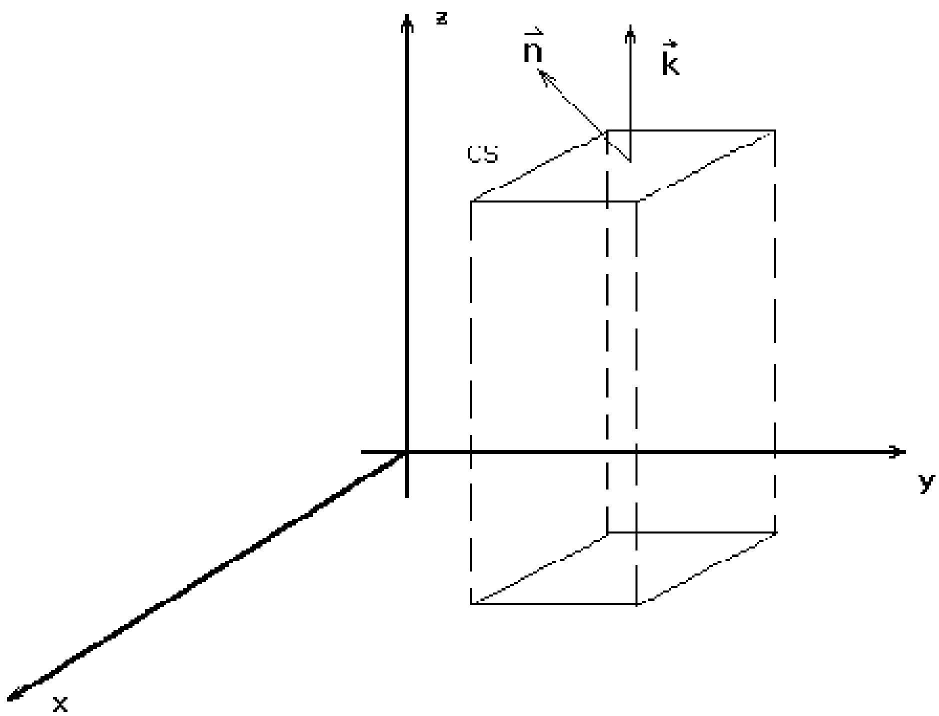


The special case $f = 1$ is of particular interest, since $\iint_S dS$ =area of curved surface S.

Remark. The case $\vartheta = 0$, i.e. $z = 0$ corresponds to finding an ordinary double integral since then S is simply a region in the x,y −plane.

Question: How does one evaluate $\iint_S f\, dS$?

Suppose S is such that it can be uniquely projected onto the x,y −plane. (We shall discuss the more general case later.) This is so if every line parallel to the z axis cuts S exactly once. As we take pieces ΔS smaller and smaller they approach flat pieces tilted with respect to the horizontal.



Now if $\vec{n}$ is the normal to $z = \vartheta(x,y)$ then $\vec{n} = -\vartheta_x \vec{i} - \vartheta_y \vec{j} + \vec{k}$. Recall if $\vec{r}$ is a vector from the origin to the surface then $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$. Then

$$\vec{n} \cdot d\vec{r} = -\vartheta_x dx - \vartheta_y dy + dz.$$

But $z = \vartheta(x,y) \Rightarrow dz = \vartheta_x dx + \vartheta_y dy \Rightarrow \vec{n} \cdot d\vec{r} = 0$. Thus $\vec{n}$ is normal to the surface.

Also $\vec{n} \cdot \vec{k} = |\vec{n}||\vec{k}|\cos\gamma$ where γ is the angle between the normal and the vertical.

$\Rightarrow$

$$\cos\gamma = \frac{\vec{n} \cdot \vec{k}}{|\vec{n}|\,|\vec{k}|} = \frac{1}{\left(1 + \vartheta_x^2 + \vartheta_y^2\right)^{\frac{1}{2}} \sqrt{1}} = \left(1 + \vartheta_x^2 + \vartheta_y^2\right)^{-\frac{1}{2}}.$$

If dA is the projection onto the horizontal plane dS of then $dA = \cos\gamma \; dS$ or

$$dS = \frac{1}{\cos\gamma} dA$$

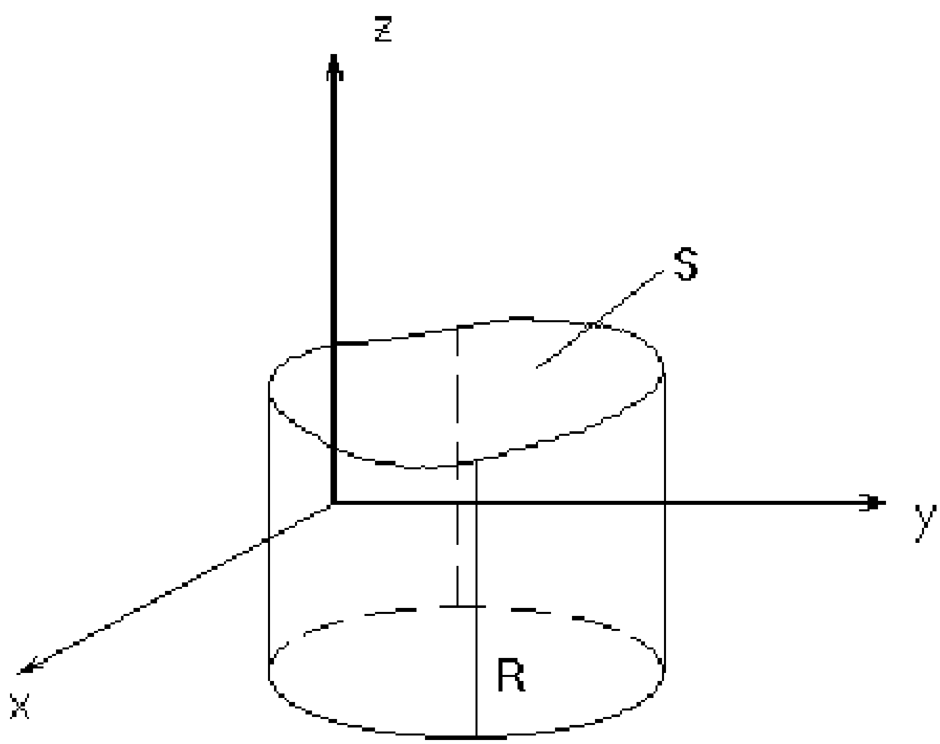
$\Rightarrow$

$$dS = \left(1 + \vartheta_x^2 + \vartheta_y^2\right)^{\frac{1}{2}} dA = \left(1 + \vartheta_x^2 + \vartheta_y^2\right)^{\frac{1}{2}} dxdy = \left(1 + \vartheta_x^2 + \vartheta_y^2\right)^{\frac{1}{2}} dydx.$$

Notice that $\vartheta = 0 \Rightarrow dA = dx\,dy = dy\,dx,$, as expected. Hence

$$\int\int_S f(x,y,z)dS = \int\int_R f(x,y,\vartheta(x,y))(1 + \vartheta_x^2 + \vartheta_y^2)^{\frac{1}{2}} dA,$$

where R is the projection of S onto the x,y –plane.



Example

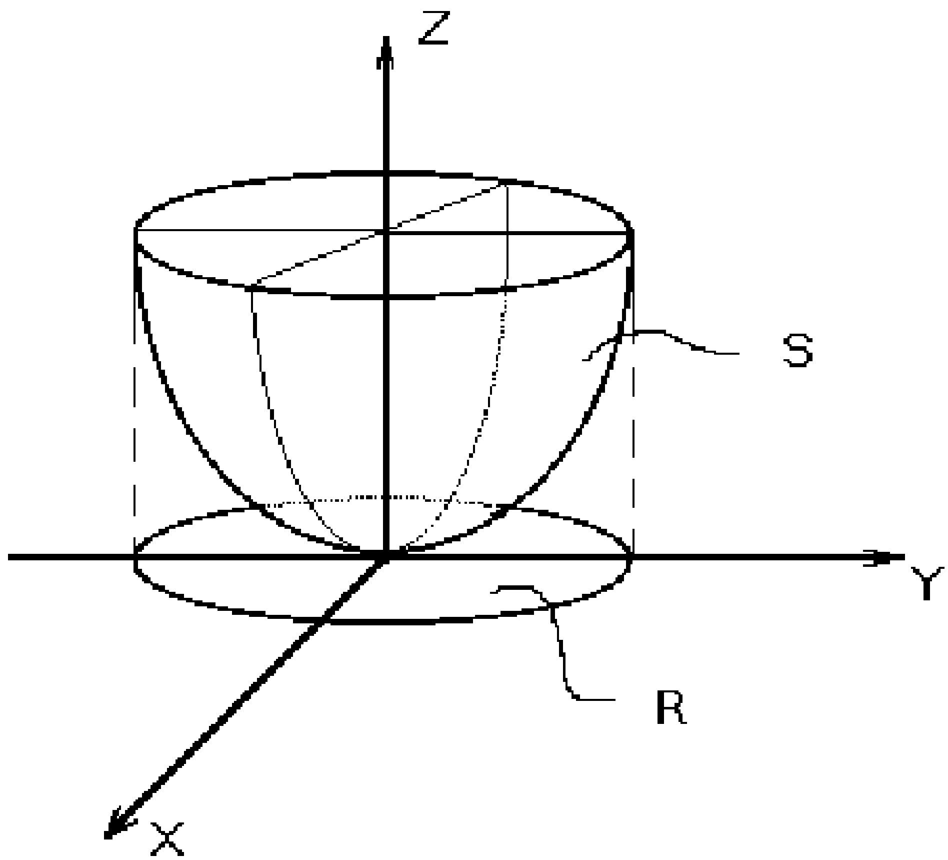
Find the surface area of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.

Solution:

The surface S projects into the interior of the circle $x^2 + y^2 = 1$. This is R. Here

$$z = \vartheta(x,y) = x^2 + y^2$$

$$\text{Surface area} = \iint_S 1 \cdot dS = \iint_R (1 + \vartheta_x^2 + \vartheta_y^2)^{\frac{1}{2}} dA$$



Here R is circle $x^2 + y^2 \leq 1$. Thus the surface area is given by

$$S = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dydx$$

To evaluate this double integral we shall use polar coordinates. Then

$$\text{Surface area} = \iint_R \sqrt{4r^2 + 1}\, rdrd\theta = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, rdrd\theta$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} \left(4r^2 + 1\right)^{\frac{3}{2}} |_0^1 d\theta = \frac{1}{12} \int_0^{2\pi} \left(5^{\frac{3}{2}-1}\right) d\theta = \frac{\pi}{6} \left(r\sqrt{5} - 1\right)$$

Check: $\int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, rdrd\theta = \frac{5}{6}\sqrt{5}\pi - \frac{1}{6}\pi$

Example

Evaluate

$$I = \iint_S x^2y^2z^2 dS$$

over the curved surface of the cone $x^2 + y^2 = z^2$ which lies between $z = 0$ and $z = 1$.

Solution:

Here $z = \vartheta(x,y) = \left(x^2 + y^2\right)^{\frac{1}{2}}$. We need z_x and z_y. Since $z^2 = x^2 + y^2 \Rightarrow 2zz_x = 2x \Rightarrow z_x = \frac{x}{z}$ and $z_y = \frac{y}{z}$.

Therefore

$$(1 + \vartheta_x^2 + \vartheta_y^2)^{\frac{1}{2}} = (1 + z_x^2 + z_y^2)^{\frac{1}{2}} = \left(1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}\right)^{\frac{1}{2}} = \left(1 + \frac{x^2 + y^2}{z^2}\right)^{\frac{1}{2}} = \sqrt{2}$$

Hence

$$I = \iint_S x^2y^2z^2 dS = \iint_R x^2y^2(x^2 + y^2)\sqrt{2}\, dxdy.$$

R is the interior of the circle $x^2 + y^2 = 1$. Again we use polar coordinates to evaluate I. Then since $z^2 = r^2$ on the cone we have

$$\begin{aligned} I &= \iint_S x^2y^2z^2 dS = \int_0^{2\pi}\int_0^1 r^2\cos^2\theta \cdot r^2\sin^2\theta \cdot r^2 \cdot \sqrt{2} \cdot rdrd\theta \\ &= \frac{\sqrt{2}}{8}\int_0^{2\pi}\left(\cos^2\theta - \cos^4\theta\right)drd\theta = \frac{\sqrt{2}}{8}\int_0^{2\pi}\left(\frac{1 + \cos 2\theta}{2} - \frac{(1 + \cos 2\theta)^2}{4}\right)d\theta \\ &= \frac{\sqrt{2}}{8}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1 + \cos 4\theta}{4}\right)d\theta = \frac{\sqrt{2}}{64}\int_0^{2x}(1 - \cos 4\theta)d\pi = \frac{\pi\sqrt{2}}{32}. \end{aligned}$$

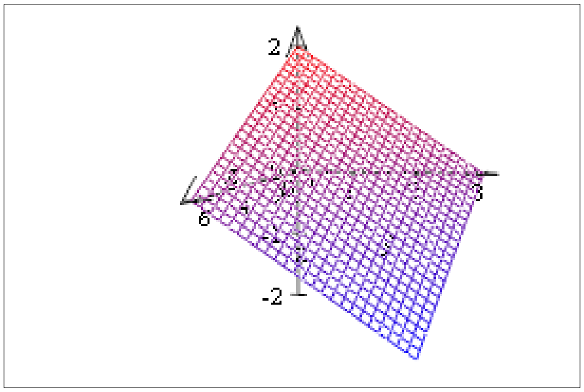
Example

Sketch the surface S given by the equation

$$x + 2y + 3z = 6$$

in the first octant.

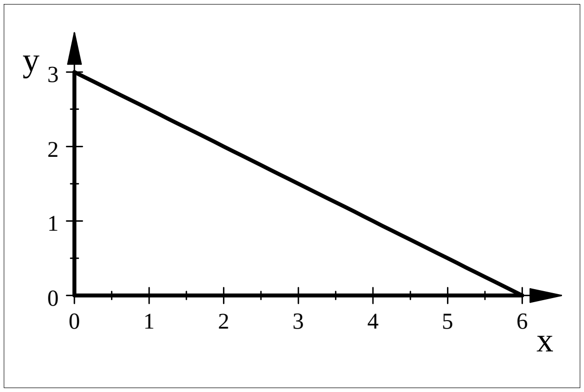
$2 - \frac{1}{3}x - \frac{2}{3}y$



Find the surface area of S.

Let S_{xy} denote the projection of S onto the x, y –plane. ThenS_{xy} is the triangle shown in the first quadrant bounded by the line $x + 2y = 6$.

$3 - \frac{x}{2}$



$$Area = \int\int_{S_{xy}} \sqrt{1 + z_x^2 + z_y^2}\, dA$$

$$= \int\int_{S_{xy}} \sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2}\, dA$$

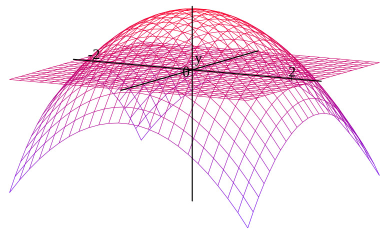
$$= \frac{\sqrt{14}}{3} \int\int_{S_{xy}} dA = \frac{\sqrt{14}}{3}\left(\frac{1}{2}\right)(6)(3) = 3\sqrt{14}$$

Example Find the surface area of the paraboloid given by $z = 4 - x^2 - y^2$ for $z \geq 0$.

Sketch the surface.

SOLUTION

The surface is a paraboloid (only the part above the x, y plane):



The formula for its surface area is $\int\int_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$.

Since $z = 4 - x^2 - y^2$,

$\int\int_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int\int_R \sqrt{1 + 4x^2 + 4y^2}\, dA$

Now the paraboloid intercepts the x, y plane, forming the circle $x^2 + y^2 = 4$.
We have to determine the limits of integration of x and y using this circle.
Introduce polar coordinates; then r goes from 0 to 2 and θ goes from 0 to 2π,
and the surface-area integral becomes

$\int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r dr d\theta = \frac{17}{6}\pi\sqrt{17} - \frac{1}{6}\pi.$