MA227 Surface Integrals

Parametrically Defined Surfaces

We discussed earlier the concept of $\int\int_S f(x,y,z)ds$ where S is given by $z = \vartheta(x,y)$. We had

$$\int\int_S fds = \int\int_R f(x,y,\vartheta(x,y))[1+\vartheta_x^2+\vartheta_y^2]^{\frac{1}{2}}dA$$

where R is the projection of S onto the x, y - plane. We shall now develop a generalization of this concept.

There are three common ways of defining a surface:

I.

$$z = \vartheta(x,y) \qquad (1)$$

as above. Here ϑ must be a single-valued, continuous function defined on a region of the plane.

II. Often surfaces are represented by equations of the form

$$F(x,y,z) = 0 \qquad (2)$$

If (x_0,y_0,z_0) is a point on such a surface, we can in many cases represent the portion of the surface near (x_0,y_0,z_0) in a form analogous to (1) by solving (2) for $x, y,$ or z in terms of the other two variables.

III. It is frequently convenient to describe a surface by a *parametric* representation.

Example:

$$x = a\sin u\cos v \qquad y = a\sin u\sin v. \qquad z = a\cos u$$

Here u and v are independent parameters. This represents a sphere whose equation is

$$x^2+y^2+z^2 = a^2$$

This equation is gotten by elimination of u and v. Note that u and v are the spherical coordinates ϕ and θ respectively.

The set of equations

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v) \qquad (3)$$

where u and v are parameters represents an arbitrary surface. This can be seen by eliminating u and v from (3), a procedure that leads to an equation of the form $F(x,y,z) = 0$ which is case II.

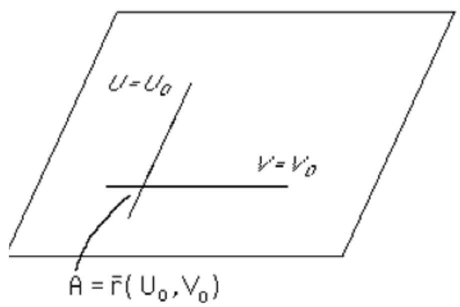
In terms of the radius vector $\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$ equation (3) for the surface may be written as

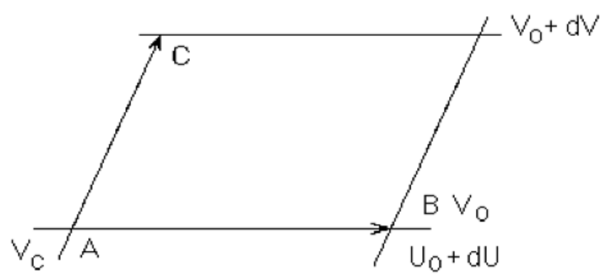
$$\vec{r} = \vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

From the parametric equations for a surface it is possible to establish a formula for *ds*, the element of surface area. In general, *ds* is obtained by calculating the area between the curves corresponding to:

$$u = u_0, \qquad u = u_0 + du, \qquad v = v_0 and v = v_0 + dv.$$

For infinitesimal areas this element will be essentially planar and have area $ds = |\vec{AB} \times \vec{AC}|$, where the vectors are the sides of the differential parallelogram shown in the diagram.





$$A = \vec{r}(u_0, v_0)$$
$$B = \vec{r}(u_0 + du, v_0) = \vec{r}(u_0, v_0) + \frac{\partial \vec{r}}{\partial u}(u_0, v_0)du + \cdots$$
$$C = \vec{r}(u_0, v_0 + dv) = \vec{r}(u_0, v_0) + \frac{\partial \vec{r}}{\partial v}(u_0, v_0)dv + \cdots$$

Thus

$$\vec{AB} = \frac{\partial \vec{r}}{\partial u}du \qquad \vec{AC} = \frac{\partial \vec{r}}{\partial v}dv$$

$\Rightarrow$

$$ds = |\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}|dudv$$

Hence, in general, we have for a surface given by

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

that

$$\int\int_S f(x,y,z)ds = \int\int_G f(u,v)|\vec{r}_u \times \vec{r}_v|dudv,$$

where G is the image of the surface S in the u,v -plane.

Suppose the surface S is given by the representation $z = \vartheta(x,y)$ (case I). Let

$$x = u, \qquad y = v \;\Rightarrow\; z = \vartheta(u,v)$$

Then

$$\vec{r}(u,v) = u\vec{i} + v\vec{j} + \vartheta(u,v)\vec{k}$$

also represents the surface. Thus

$$\vec{r}_u = \vec{i} + \vartheta_u\vec{k}; \qquad \vec{r}_v = \vec{j} + \vartheta_v\vec{k};$$

and

$$\vec{r_u} \times \vec{r_v} = \vec{k} - \vartheta_u\vec{i} - \vartheta_v\vec{j}$$

so

$$ds = |\vec{r}_u \times \vec{r_v}|dudv = [1 + \vartheta_u^2 + \vartheta_v^2]^{\frac{1}{2}}dudv$$

But since $u = x,\ v = y$ we get

$$ds = [1 + \vartheta_x^2 + \vartheta_y^2]^{\frac{1}{2}}dxdy$$

as before.

Example

We shall find the surface area of a sphere of radius a centered at the origin. The equation of the sphere is

$$x^2 + y^2 + z^2 = a^2$$

In spherical coordinates the sphere is given by

$$x = a\sin u\cos v \qquad y = a\sin u\sin v \qquad z = a\cos u$$

$\Rightarrow$

$$\vec{r} = a\sin u\cos v\vec{i} + a\sin u\sin v\vec{j} + a\cos u\vec{k}$$

Hence

$$\vec{r_u} \times \vec{r_v} = a^2(\sin^2 u \cos v\,\vec{i} + \sin^2 u \sin v\,\vec{j} + \sin u \cos u\,\vec{k})$$

$\Rightarrow$

$$|\vec{r_u} \times \vec{r_v}| = a^2 \sin u$$

and

$$ds = a^2 \sin u\,du\,dv$$

Thus

$$\int\int_S ds = \int\int_S a^2 \sin u\,du\,dv = \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi\,d\phi\,d\theta = 4\pi a^2 = \text{surface area of a sphere.}$$

Surface Elements

Suppose that R is a closed rectangular region in the u,v – plane, where $a \leq u \leq b$, $c \leq v \leq d$. Then the equations $x = x(u,v),\ \ y = y(u,v),\ \ z = z(u,v),$ where x,y,z are continuous, define a set S which is part of a surface in x,y,z – space. If the functions x,y,z are also 1-1, i.e. distinct points of R are not mapped into the same point of S, then the points of S in x,y,z – space comprise a *simple surface element*. A simple surface element may be thought of as any configuration which may be obtained from a rectangular plane region by continuous deformation (bending, twisting, stretching, shrinking) without tearing and without bringing together any points which were originally distinct.

If S is a simple surface element corresponding to a rectangular region R in the u,v – plane, the points of S which correspond to the boundary of R form what is called the boundary S. Other points of S are called interior points.

All surfaces may be thought of as being built up out of simple surface elements by matching together portions of the edges of the elements. The *boundary* of a surface consists of the *unmatched* edges of its surface elements. If there are no unmatched edges, there is no boundary. For example, a hemisphere has a boundary consisting of its equatorial rim. An entire sphere, an ellipsoid, and the surface of a cube are examples of surfaces without boundary.

A surface is smooth if the functions which parametrize it are continuously differentiable. If a surface is smooth and has no boundary, it is called a *smooth surface without boundary*. If a surface is given by $F(x,y,z) = 0$, then the surface is smooth without boundary if $\nabla F \neq 0$ for all x,y,z on the surface.

Example: Consider the surface

$$F(x,y,z) = 4x^2 + 9y^2 - 2z^2 - 8 = 0$$

Then

$$\nabla F = 8x\,\vec{i} + 18y\,\vec{j} - 4z\,\vec{k}$$

and $\nabla F = 0 \ \Rightarrow\ x = y = z = 0$. But $(0,0,0)$ is not on this surface. $\Rightarrow F$ is smooth without boundary.

Surface Integrals

Example

Evaluate $\int\int_S f(x,y,z)ds$ where $f = x^2$ and S is the part of the cone

$$z^2 = x^2 + y^2$$

between the planes $z = 1$ and $z = 2$.

We shall use spherical coordinates

$$x = \rho\sin\phi\cos\theta\ , y = \rho\sin\phi\sin\theta, z = \rho\cos\phi.$$

In spherical coordinates the equation of the cone is $\phi = \frac{\pi}{4}$. Letting $u = \theta$, $v = \rho$ $\Rightarrow$ we have for x, y, and z on the surface of the cone that

$$x(u,v) = x(\theta,\rho) = \frac{\sqrt{2}}{2}\rho\cos\theta;\ y(u,v) = y(\theta,\rho) = \frac{\sqrt{2}}{2}\rho\sin\theta;\ z = \frac{\sqrt{2}}{2}\rho$$

where $0 \le \theta \le 2\pi$ and $1 \le z \le 2 \ \Rightarrow\ \sqrt{2} \le \rho \le 2\sqrt{2}$

$\Rightarrow$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \frac{\sqrt{2}}{2}\rho\cos\theta\vec{i} + \frac{\sqrt{2}}{2}\rho\sin\theta\vec{j} + \frac{\sqrt{2}}{2}\rho\vec{k}$$

$\Rightarrow$

$$\vec{r_u} = \vec{r_\theta} = -\frac{\sqrt{2}}{2}\rho\sin\theta\vec{i} + \frac{\sqrt{2}}{2}\rho\cos\theta\vec{j}$$

$\Rightarrow$

$$\vec{r_v} = \vec{r_\rho} = \frac{\sqrt{2}}{2}\cos\theta\vec{i} + \frac{\sqrt{2}}{2}\sin\theta\vec{j} + \frac{\sqrt{2}}{2}\vec{k}$$

$\Rightarrow$

$$\vec{r_\theta} \times \vec{r_\rho} = \frac{1}{2}\rho[\cos\theta\vec{i} + \sin\theta\vec{j} - \vec{k}] \quad \text{and} \quad |\vec{r_\theta} \times \vec{r_\rho}| = \frac{\sqrt{2}}{2}\rho$$

$$\int\int_S x^2 ds = \int_{\sqrt{2}}^{2\sqrt{2}}\int_0^{2\pi}\left(\frac{1}{2}\rho^2\cos^2\theta\right)\left(\frac{\sqrt{2}}{2}\rho\right)d\theta d\rho$$

$$= \frac{\sqrt{2}}{8}\int_{\sqrt{2}}^{2\sqrt{2}}\rho^3\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} d\rho = \frac{15}{4}\sqrt{2\pi}$$

Example

Evaluate the integral of

$$f(x,y,z) = \left(x^2 + y^2\right)z$$

over the upper half of the sphere of radius 1 centered at the origin.
We shall use spherical coordinates to parametrize the hemisphere. Since $\rho = 1$, we have

$$x(\phi,\theta) = \sin\phi\cos\theta, \quad y(\phi,\theta) = \sin\phi\sin\theta, \quad z(\phi,\theta) = \cos\phi$$

Thus

$$\vec{r}(\phi,\theta) = \sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}$$

where $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$.
Then

$$\vec{r}_\theta(\phi,\theta) = -\sin\phi\sin\theta\vec{i} + \sin\phi\cos\theta\vec{j}$$
$$\vec{r}_\phi(\phi,\theta) = \cos\phi\cos\theta\vec{i} + \cos\phi\sin\theta\vec{j} - \sin\phi\vec{k}$$

Hence

$$|\vec{r}_\theta \times \vec{r}_\rho| = \sin\phi$$

Therefore

$$\iint_S f(x,y,z)ds = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta\right)\cos\phi\sin\phi d\phi d\theta$$
$$= \frac{\pi}{2}$$

Remark: Very often one is interested in an integral of the form

$$\int\int_S \vec{F}\cdot\vec{n}ds$$

where $\vec{n}$ is a unit normal (perpendicular) vector to the surface S pointing in the outward direction. From the discussion above it follows that the vectors $\overrightarrow{r_u}$ and $\overrightarrow{r_v}$ are both in the "plane" of the surface. Thus $\overrightarrow{r_u} \times \overrightarrow{r_v}$ is $\perp$ to the surface S. Hence

$$\pm\frac{\overrightarrow{r_u}\times\overrightarrow{r_v}}{|\overrightarrow{r_u}\times\overrightarrow{r_v}|} \text{ is a unit normal.}$$

We choose the appropriate sign (either + or -) which makes this unit vector outward. One can select an appropriate point on the surface and see if $+\frac{\overrightarrow{r_u}\times\overrightarrow{r_v}}{|\overrightarrow{r_u}\times\overrightarrow{r_v}|}$ is inward or outward.

If it is inward, then use $-\frac{\overrightarrow{r_u}\times\overrightarrow{r_v}}{|\overrightarrow{r_u}\times\overrightarrow{r_v}|}$.

Note that

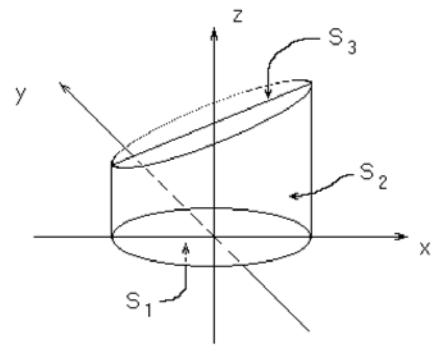
$$\int\int_S \vec{F}\cdot\vec{n}ds = \int\int_D \vec{F}\cdot\left(\frac{\overrightarrow{r_u}\times\overrightarrow{r_v}}{|\overrightarrow{r_u}\times\overrightarrow{r_v}|}\right)(|\overrightarrow{r_u}\times\overrightarrow{r_v}|)dudv$$
$$= \int\int_D \vec{F}\cdot(\overrightarrow{r_u}\times\overrightarrow{r_v})dudv$$

Thus, unless one is asked specifically for the unit vector $\vec{n}$, it is not necessary to calculate $|\vec{r_u} \times \vec{r_v}|$.

Example

Let R be the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = x + 2$. Let S be the entire boundary of R. Find the value of $\int\int_S \vec{F} \cdot \vec{n} ds$ where $\vec{n}$ is the outward directed unit normal on S and

$$\vec{F} = 2x\vec{i} - 3y\vec{j} + z\vec{k}.$$



Now S is composed of $S_1, S_2,$ and S_3.

On S_1 $\vec{n} = -\vec{k} \Rightarrow \vec{F} \cdot \vec{n} = -z$. But on $z = 0$ on $S_1 \Rightarrow \vec{F} \cdot \vec{n} = 0 \Rightarrow$

$$\int\int_{S_1} \vec{F} \cdot \vec{n} ds = 0$$

On S_3 $z = x + 2 \Rightarrow$ we parametrize as $x = u \qquad y = v \qquad z = u + 2$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = u\vec{i} + v\vec{j} + (u + 2)\vec{k}$$

$$\vec{r_u} = \vec{i} + \vec{k} \qquad \vec{r_v} = \vec{j} \qquad \Rightarrow \vec{r_u} \times \vec{r_v} = \vec{k} - \vec{i}$$

This is outer

$\Rightarrow$

$$\vec{F} \cdot \left(\vec{r_u} \times \vec{r_v}\right) = -2x + z = -2u + u + 2 = -u + 2$$

so that

$$\int\int_{S_3} \vec{F} \cdot \vec{n} ds = \int\int_G (-u + 2) dudv$$

Where G is the projection of S_3 in the $u, v -$ plane. But since $u = x, \ v = y$ and the plane $z = x + 2$ slices the cylinder $x^2 + y^2 = 1$, we see that G is the interior of the circle $x^2 + y^2 \leq 1$. Thus on S_3 we have

$$\iint_{S_3} \vec{F} \cdot \vec{n} ds = \iint_{x^2+y^2 \le 1} (-x+2) dA$$
$$= -\int_0^{2\pi} \int_0^1 r\cos\theta r dr d\theta + 2 \iint_{x^2+y^2\le 1} dA$$
$$= -\frac{1}{3} \int_0^{2\pi} \cos\theta d\theta + 2\pi = 2\pi$$

On S_2 we shall use cylindrical coordinates $x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$
Since our cylinder is $x^2 + y^2 = 1 \Rightarrow r = 1 \Rightarrow$

$$\vec{r} = \cos\theta\overrightarrow{i + }\sin\theta\vec{j} + z\vec{k}$$

where $0 \le z \le x + 2 = \cos\theta + 2$, and $0 \le \theta \le 2\pi$.
Taking $u = \theta \qquad v = z$ here, we have

$$\vec{r_\theta} = -\sin\theta\vec{i} + \cos\theta\vec{j} \qquad \vec{r_z} = \vec{k}$$

$\Rightarrow$

$$\vec{r_\theta} \times \vec{r_z} = \cos\theta\vec{i} + \sin\theta\vec{j} \Rightarrow |\vec{r_\theta} \times \vec{r_z}| = 1$$

Thus we may use $\vec{N} = \cos\theta\vec{i} + \sin\theta\vec{j}$. This vector is outward, since $\theta = 0°$ gives $\vec{n} = \vec{i}$.

$$\vec{F} \cdot \vec{N} = (2\cos\theta\vec{i} - 3\sin\theta\vec{j} + z\vec{k}) \cdot \vec{N} = 2\cos^2\theta - 3\sin^2\theta$$

Hence

$$\iint_{S_2} \vec{F} \cdot \vec{n} ds = \int_0^{2\pi} \int_0^{2+\cos\theta} (2\cos^2\theta - 3\sin^2\theta) dz d\theta$$
$$= \int_0^{2\pi} \int_0^{2+\cos\theta} (2 - 5\sin^2\theta) dz d\theta = -2\pi$$

Thus we have finally

$$\iint_S \vec{F} \cdot \vec{n} ds = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3} \right) \vec{F} \cdot \vec{n} ds = 0 + 2\pi - 2\pi = 0.$$

Remark: Stewart uses the notation

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} ds$$

for the surface integral of $\vec{F}$ over a surface S. He also calls the $\iint_S \vec{F} \cdot d\vec{S}$ the **flux** of $\vec{F}$ over S.

Example

Parametrize the surface S that is the part of the paraboloid

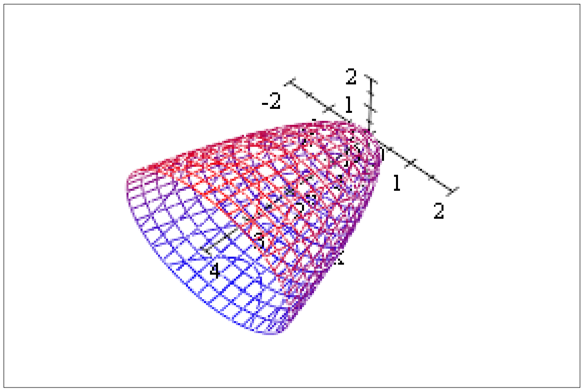
$$x = y^2 + z^2$$

that lies between the planes $x = 4$ and $x = 0$, and give an expression for

$$\iint_S x ds$$

Sketch the surface S.

Solution:



Let

$$y = u\sin v,\ z = u\cos v,\ \ x = u^2$$

where $0 \le v \le 2\pi$, and $0 \le x \le 4$ implies $0 \le u \le 2$.

$$\vec{r}(u,v) = u^2\vec{i} + u\sin v\vec{j} + u\cos v\vec{k}$$

so

$$\vec{r}_u = 2u\vec{i} + \sin v\vec{j} + \cos v\vec{k}$$
$$\vec{r}_v = u\cos v\vec{j} - u\sin v\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & \sin v & \cos v \\ 0 & u\cos v & -u\sin v \end{vmatrix}$$
$$= 2\vec{j}u^2\sin v + 2\vec{k}u^2\cos v + \vec{i}\left(-\left(\sin^2 v\right)u - \left(\cos^2 v\right)u\right)$$
$$= -u\vec{i} + 2u^2(\sin v)\vec{j} + -2u^2\cos v\vec{k}$$

:

Thus

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4u^4\cos^2 v + 4u^4\sin^2 v + u^2}$$
$$= u\sqrt{4u^2+1}$$

Thus

$$\iint_S x ds = \iint_{0\le y^2+z^2\le 4} x|\vec{r}_u \times \vec{r}_v| dudv$$
$$= \int_0^{2\pi}\int_0^2 u^2\left(u\sqrt{4u^2+1}\right)dudv = \frac{391}{60}\sqrt{17}\pi + \frac{1}{60}\pi$$

Example 5 page 956 in Stewart

Evaluate

$$\iint_S \vec{F} \cdot d\vec{S}$$

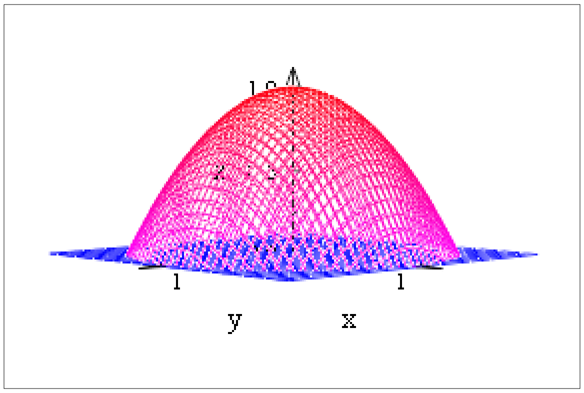
where

$$\vec{F}(x,y,z) = y\vec{i} + x\vec{j} + z\vec{k}$$

and S if the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution:

The graph of the surface is shown below.



The closed surface S consists of a parabolic top surface S_1 and a circular bottom surface S_2 : $x^2 + y^2 \leq 1,\ z = 0$.

We may parametrize the surface S_1 as

$$x = u, \quad y = v, \quad z = 1 - u^2 - v^2$$

Then

$$\vec{r}(u,v) = u\vec{i} + v\vec{j} + \left(1 - u^2 - v^2\right)\vec{k}$$

or

$$\vec{r}(u,v) = \left(u, v, 1 - u^2 - v^2\right)$$

Thus

$$\frac{\partial\vec{r}(u,v)}{\partial u} = (1, 0, -2u)$$
$$\frac{\partial\vec{r}(u,v)}{\partial v} = (0, 1, -2v)$$

Hence

$$\vec{r}_u \times \vec{r}_v = (1,0,-2u) \times (0,1,-2v) = (2u, 2v, 1) = 2u\vec{i} + 2v\vec{j} + \vec{k}$$

The projection of S_1 onto the u,v –plane, which is this case is the x,y –plane, since $x = u$ and $y = v$, is the circle $D :\ x^2 + y^2 \leq 1$. Thus using x and y instead of u and v we have

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \left(y\vec{i} + x\vec{j} + z\vec{k}\right) \cdot \left(2x\vec{i} + 2y\vec{j} + \vec{k}\right) dA$$
$$= \iint_D \left(y\vec{i} + x\vec{j} + \left(1 - x^2 - y^2\right)\vec{k}\right) \cdot \left(2x\vec{i} + 2y\vec{j} + \vec{k}\right) dA$$
$$= \iint_{x^2+y^2\le 1} \left(4xy + 1 - x^2 - y^2\right) dA$$

Since we are integrating over a circle of radius 1 centered at the origin, we switch to polar coordinates and have

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 \left(4r^2 \cos\theta \sin\theta + 1 - r^2\right) r dr d\theta = \frac{1}{2}\pi$$

Now on S_2 $\vec{n} = -\vec{k}$, $z = 0$, and $\vec{F}(x,y,z) = y\vec{i} + x\vec{j}$ so that $\vec{F} \cdot \vec{n} = 0$ and

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = 0$$

Finally

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$$
$$= \frac{1}{2}\pi + 0 = \frac{\pi}{2}$$

Example

Let S be the surface of the solid cylinder T bounded by $z = 0$ and $z = 3$ and $x^2 + y^2 = 4$. Evaluate

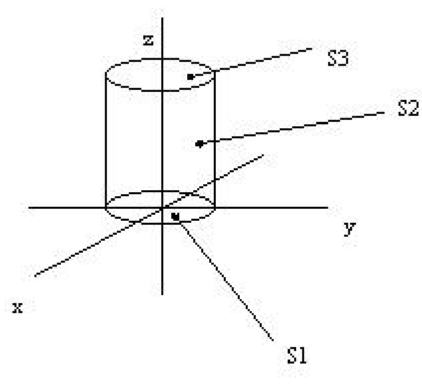
$$\int\int_S \vec{F} \cdot \vec{n} dS,$$

where

$$\vec{F} = (x^2 + y^2 + z^2)(x\vec{i} + y\vec{j} + z\vec{k})$$

and $\vec{n}$ is the outward unit normal. Sketch the surface.

SOLUTION



S is composed of $S_1, S_2,$ and S_3.

On S_1 $\vec{n} = -\vec{k} \Rightarrow$

$$\vec{F} \cdot \vec{n} = -z(x^2 + y^2 + z^2).$$

But $z = 0$ on $S_1 \Rightarrow \vec{F} \cdot \vec{n} = 0 \Rightarrow$

$$\int\int_{S_1} \vec{F} \cdot \vec{n} ds = 0.$$

On S_3 $z = 3$, $\vec{n} = +\vec{k} \Rightarrow$

$$\vec{F} \cdot \vec{n} = +z(x^2 + y^2 + z^2) = 3(x^2 + y^2 + 9) = 3x^2 + 3y^2 + 27.$$

Since S_3 is a disk of radius 2 we introduce polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $ds = rdrd\theta$ and $r^2 = x^2 + y^2$ so

$$\int\int_{S_3} \vec{F} \cdot \vec{n} ds = \int\int_{S_3} (3x^2 + 3y^2 + 27) dxdy = 3\int_0^{2\pi}\int_0^2 (r^2 + 9) rdrd\theta = 132\pi$$

On S_2 we shall use cylindrical coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

Since our cylinder is $x^2 + y^2 = 4 \Rightarrow r = 2 \Rightarrow$

$$\vec{r} = 2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k} \qquad \text{where} \quad 0 \le z \le 3.$$

Taking $u = \theta$ $\quad v = z$ here, we have

$$\vec{r_\theta} = -2\sin\theta\vec{i} + 2\cos\theta\vec{j} \qquad \vec{r_z} = \vec{k}$$

$\Rightarrow$

$$\vec{r_\theta} \times \vec{r_z} = 2\cos\theta\vec{i} + 2\sin\theta\vec{j}$$

Thus we may use $\vec{N} = 2\cos\theta\vec{i} + 2\sin\theta\vec{j}$ for a normal, since this is outward.

$$\vec{F} = (x^2 + y^2 + z^2)(x\vec{i} + y\vec{j} + z\vec{k})$$

so

$$\vec{F} = (4\cos^2\theta + 4\sin^2\theta + z^2)(2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k})$$

Then

$$\vec{F} \cdot \vec{N} = 2(4 + z^2)(2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k}) \cdot (\cos\theta\vec{i} + \sin\theta\vec{j})$$
$$= 2(4 + z^2)(2\cos^2\theta + 2\sin^2\theta) = 4\left(4 + z^2\right)$$

Hence

$$\int\int_{S_2} \vec{F} \cdot \vec{n} ds = 4\int_0^{2\pi}\int_0^3 (4 + z^2) dzd\theta = 168\pi$$

Thus we have finally

$$\int\int_S \vec{F} \cdot \vec{n} ds = \left(\int\int_{S_1} + \int\int_{S_2} + \int\int_{S_3}\right)\vec{F} \cdot \vec{n} ds = 0 + 132\pi + 168\pi = 300\pi.$$

Stokes' Theorem and the Divergence Theorem

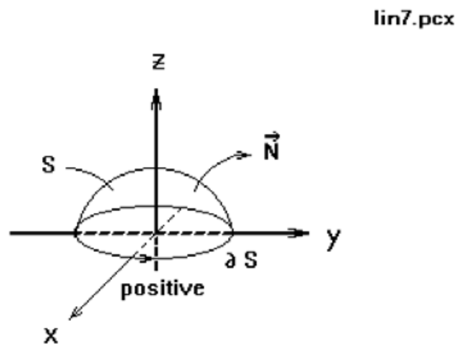
Stokes' Theorem:

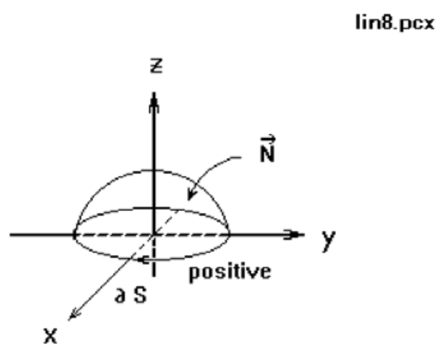
Let S be a regular surface bounded by a closed curve denoted by ∂S (boundary of S). Let $\vec{F}$ and *curl* $\vec{F}$ be continuous over S. Then

$$\iint_S \text{curl}\,\vec{F} \cdot \vec{n}ds = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}ds = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

Here the direction of integration around ∂S is positive with respect to the side of S on which the normal $\vec{n}$ is drawn.

Remark:

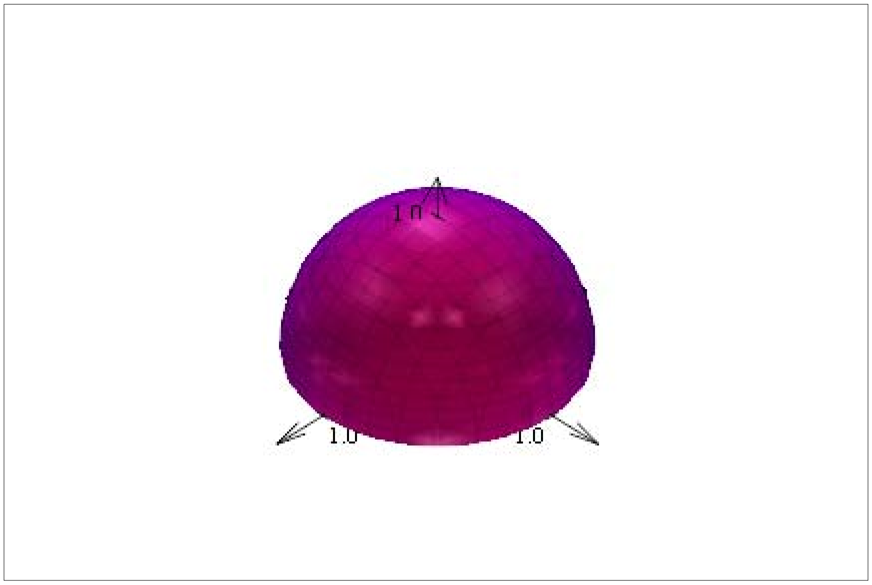




Example

Verify Stokes' Theorem when $\vec{F} = y\vec{i} + 3z\vec{j} + 3x\vec{k}$ and S is the hemispheric surface $z = \sqrt{1 - x^2 - y^2}$.

$x^2 + y^2 + z^2 - 1 = 0$



We shall use the outward normal $\vec{n}$. We calculate $\oint_{\partial S} \vec{F} \cdot d\vec{r}$ first. Now ∂S is the circle $x^2 + y^2 = 1$, $z = 0$. We parametrize this as

$$x = \cos t, \qquad y = \sin t, \qquad z = 0 \qquad 0 \le t \le 2\pi$$

$$\vec{F} = \sin t\,\vec{i} + 0\vec{j} + 3\cos t\,\vec{k}$$

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k} = \cos t\,\vec{i} + \sin t\,\vec{j} + 0\vec{k} \;\Rightarrow\; \vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

Thus

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\sin^2 t dt = -\pi.$$

Now consider

$$\int\int_S \operatorname{curl} \vec{F} \cdot \vec{n} ds.$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 3z & 3x \end{vmatrix} = -3\vec{i} - 3\vec{j} - \vec{k}$$

S is the surface $x^2 + y^2 + z^2 = 1 \qquad z \ge 0$. In spherical coordinates $\rho = 1 \Rightarrow$

$$x = \sin\phi\cos\theta, \qquad y = \sin\phi\sin\theta, \qquad z = \cos\phi$$

Let $u = \phi \qquad v = \theta$ and therefore

$$\vec{r}(u,v) = \sin u \cos v \vec{i} + \sin u \sin v \vec{j} + \cos u \vec{k}$$

so that

$$\vec{r_u} \times \vec{r_v} = \sin^2 u \cos v \vec{i} + \sin^2 u \sin v \vec{j} + \sin u \cos u \vec{k}$$

At $\phi = \frac{\pi}{2}, \qquad \theta = 0,$ i.e. $u = \frac{\pi}{2} \qquad v = 0 \Rightarrow$

$\vec{r_u} \times \vec{r_v} = \vec{i}$ which is outward. Let $\vec{N} = \vec{r_u} \times \vec{r_v}$ is outward.

Then

$$\operatorname{curl} \vec{F} \cdot \vec{N} = -3\sin^2 u \cos v - 3\sin^2 u \sin v - \sin u \cos u$$

$$\begin{aligned}
\int\int_S \operatorname{curl} \vec{F} \cdot \vec{N} ds &= -\int_0^{2\pi} \int_0^{\frac{\pi}{2}} (3\sin^2 u \cos v + 3\sin^2 u \sin v + \sin u \cos u) du dv \\
&= -3\int_0^{2\pi} \int_0^{\frac{\pi}{2}} (\cos v + \sin v) \sin^2 u du dv - \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos u \sin u du dv \\
&= -\frac{3}{2} \int_0^{2\pi} (\cos v + \sin v)[u - \frac{\sin 2u}{2}]_0^{\frac{\pi}{2}} dv - \frac{1}{2} \int_0^{2\pi} dv \\
&= -\frac{3}{2}(\frac{\pi}{2}) \int_0^{2\pi} [\cos v + \sin v] dv - \pi = -\frac{3\pi}{4}[-\sin v + \cos v]_0^{\frac{\pi}{2}} - \pi = -\pi
\end{aligned}$$

as before.

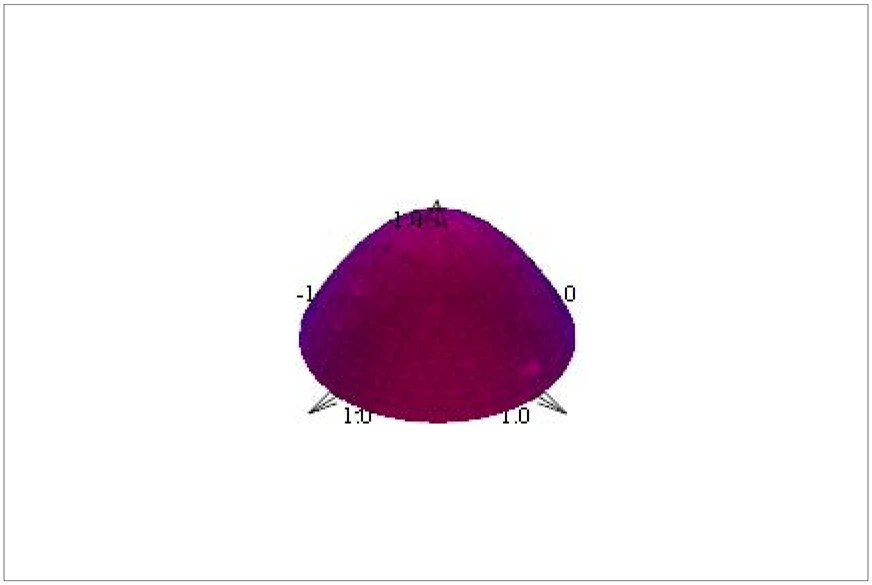
Example

Verify Stokes' Theorem is true for the vector field

$$\vec{F}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

and S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the x,y –plane and S has upward orientation. Sketch S.

$1 - x^2 - y^2 - z = 0$



We must show

$$\iint_S \text{curl}\vec{F} \cdot \vec{n}ds = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \nabla \times \left(x^2, y^2, z^2\right) = (0,0,0)$$

Thus

$$\iint_S curl\vec{F} \cdot \vec{n}ds = 0$$

For the line integral we parametrize the boundary of S, namely the circle $x^2 + y^2 = 1$ in the x,y –plane, as

$$x = \cos t,\ \ y = \sin t,\quad z = 0\quad 0 \le t \le 2\pi$$

so

$$\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 0\vec{k}$$
$$\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j}$$
$$\vec{F}(t) = \cos^2 t\vec{i} + \sin^2 t\vec{j} + 0\vec{k}$$

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(-\cos^2 t \sin t + \sin^2 t \cos t\right)dt$$
$$= \left[\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3} \right]_0^{2\pi} = 0$$

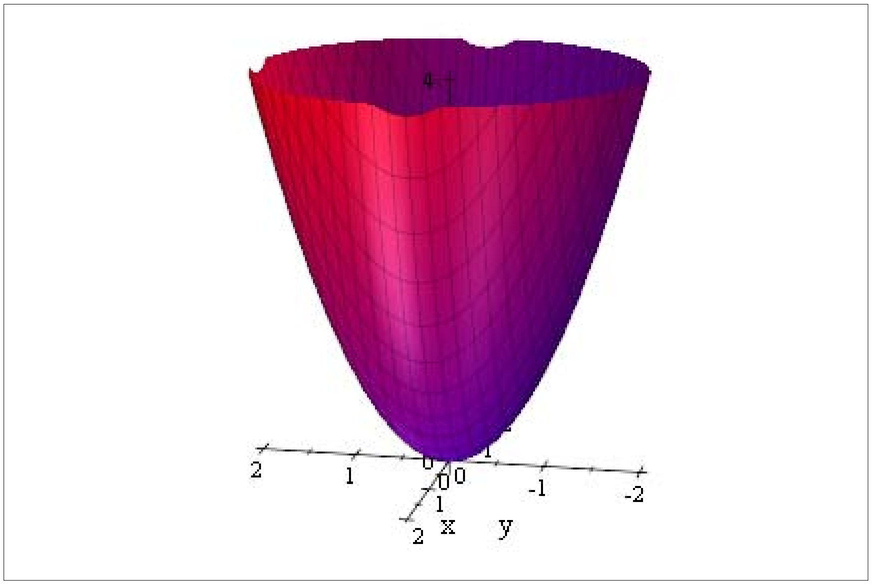
Example Evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}dS,$ where

$$\vec{F}(x,y,z) = 3z\vec{i} + 5x\vec{j} - 2y\vec{k}$$

and S is the part of the parabolic surface $z = x^2 + y^2$ that lies below the plane $z = 4$ and whose orientation is given by the upward unit normal vector.

Solution: The surface is shown below:

$x^2 + y^2$



We use Stokes' Theorem to evaluate this integral where C is the circle $x^2 + y^2 = 4,\ z = 4,\ 0 \le t \le 2\pi$
Then

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}ds = \oint_{x^2+y^2=4} \vec{F} \cdot d\vec{r}$$

C may be parametrized as $x = 2\cos t, y = 2\sin t,\ z = 4,$ so $\vec{r} = 2\cos t\vec{i} + 2\sin t\vec{j} + 4\vec{k}$ and

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}ds = \oint_{x^2+y^2=4} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(t) \cdot \vec{r}'(t)dt = \int_0^{2\pi} \left(3(4)\vec{i} + 10\cos t\vec{j} - 4\sin t\vec{k}\right) \cdot \left(-2\sin t\vec{i} + 2\cos t\vec{j} + 0\vec{k}\right)dt$$

$$= \int_0^{2\pi} \left(-24\sin t + 20\cos^2 t\right)dt = \int_0^{2\pi} (-24\sin t + 10 + 10\cos 2t)dt = 20\pi$$

Alternatively, we can directly compute the surface integral. First we calculate the integrand.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 5x & -2y \end{vmatrix} = -2\vec{i} + 3\vec{j} + 5\vec{k}$$

For the surface, we use x and y as parameters and have

$$\vec{r} = \left\langle x, y, x^2 + y^2 \right\rangle, 0 \leqq x^2 + y^2 \leqq 4$$
$$\vec{r}_x = \langle 1, 0, 2x \rangle$$
$$\vec{r}_y = \langle 0, 1, 2y \rangle$$
$$\vec{r}_x \times \vec{r}_y = \langle -2x, -2y, 1 \rangle$$

Then

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}dS = \iint_D \langle -2,3,5\rangle \cdot \langle -2x,-2y,1\rangle dA_{xy}$$
$$= \iint_D (4x-6y+5)dA_{xy}$$
$$= \int_0^{2\pi}\int_0^2 (4r\cos\theta - 6r\sin\theta + 5)rdrd\theta$$
$$= 20\pi.$$

Here, the integral is changed into polar coordinates, since the region of integration is the disc $0 \leqq x^2+y^2 \leqq 4$.

The Divergence Theorem (Gauss's Theorem)

Remark: We shall call a surface *positively oriented* if the normal $\vec{N}$ is an outer normal; otherwise, S is *negatively oriented*.

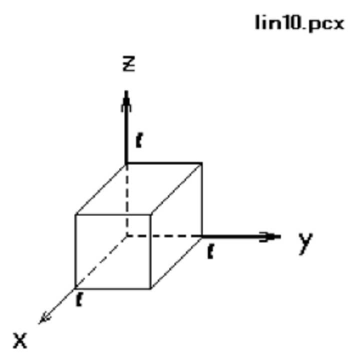
Theorem: Suppose S is a regular, positively oriented, closed surface, and that $\vec{F}$ and div$\vec{F}$ are continuous over S and the region V is enclosed by S.

Then

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_S \vec{F}\cdot\vec{n}ds = \int\int\int_V \operatorname{div}\vec{F}dV = \int\int\int_V \nabla\cdot\vec{F}dV$$

where $\vec{n}$ is the *outward* unit normal to S.
Note: $\vec{n}$ *must* be outward.

Example: Check the validity of the divergence theorem if $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$, where V is the volume of the cube $0 \le x,y,z \le \ell$.



$$div\vec{F} = 1+1+1 = 3.$$

Hence

$$\int\int\int_V div\vec{F}dV = 3\int\int\int_V dV = 3V = 3\ell^3$$

Now we must calculate $\int\int_S \vec{F}\cdot\vec{n}ds$ over all six faces of the cube. On $x = \ell$ we use

$$\vec{n} = \vec{i} \Rightarrow \vec{F}\cdot\vec{n} = (\ell\vec{i} + y\vec{j} + z\vec{k})\cdot\vec{i} = \ell$$

$$\int\int_{\text{Face } x=\ell} \vec{F}\cdot\vec{n}ds = \ell\int\int_{\text{Face } x=\ell} ds = \ell \times (\text{area of face}) = l^3$$

On $x = 0$ $\quad \vec{F} = y\vec{j} + z\vec{k}$ we may take $\vec{n} = -\vec{i}$. Thus $\vec{F}\cdot\vec{n} = 0$ $\quad$ Thus the contribution from this face is 0.

We get similarly for $y = \ell$, $\int\int_{\text{Face } y=l} \vec{F}\cdot\vec{n}ds = \ell^3$, whereas for $y = 0, \int\int_{\text{Face } y=0} \vec{F}\cdot\vec{n}ds = 0$.

And for the face $z = \ell$, $\int\int_{\text{Face } z=l} \vec{F}\cdot\vec{n}ds = \ell^3$ and on $z = 0, \int\int_{\text{Face } z=0} \vec{F}\cdot\vec{n}ds = 0$, .

Finally we have

$$\int\int_S \vec{F}\cdot\vec{n}ds = \ell^3 + \ell^3 + \ell^3 = 3\ell^3$$

where S is the entire surface of the cube.

Example

Verify Gauss's Divergence theorem, namely

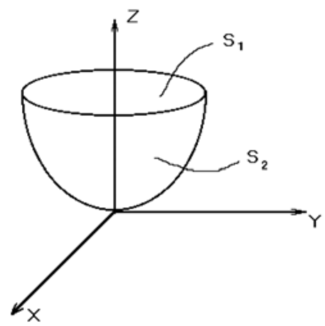
$$\int\int_S \vec{F}\cdot\vec{n}ds = \int\int\int_V \text{div}\vec{F}dV$$

where $\vec{F} = (x - y + z)\vec{i} + 2x\vec{j} + \vec{k}$ and S is the closed parabolic bowl consisting of the two pieces

$$S_1 : \text{the circle} \quad x^2 + y^2 \leq 1, \qquad z = 1,$$

and

$$S_2 : z = x^2 + y^2; \qquad x^2 + y^2 \leq 1$$



Thus S_2 is the bowl proper and S_1 is the circular cap on top. Since $\vec{\nabla} \cdot \vec{F} = 1 \Rightarrow$

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F} dv = \iiint\limits_V 1 dv = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \int_{x^2+y^2}^{1} dzdydx$$
$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2) dydx = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta$$
$$= \int_0^{2\pi} (\frac{r^2}{2} - \frac{r^4}{4})|_0^1 d\theta = \frac{\pi}{2}$$

We now evaluate

$$\iint_S \vec{F} \cdot \vec{n} ds = \left(\iint_{S_1} + \iint_{S_2} \right) \vec{F} \cdot \vec{n} ds$$

On S_2 we use cylindrical coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad z = z$$

$\Rightarrow$

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad z = x^2 + y^2 = r^2,$$

Let $r = u, \quad \theta = v \quad \Rightarrow \quad x = u\cos v, \quad y = u\sin v, \quad z = u^2 \quad 0 \le u \le 1 \quad 0 \le v \le 2\pi$

$\Rightarrow$

$$\vec{r}(u,v) = u\cos v\vec{i} + u\sin v\vec{j} + u^2\vec{k}$$

$$\vec{r_u} = \cos v\vec{i} \sin v\vec{j} + 2u\vec{k}$$
$$\vec{r_v} = -u\sin v\overrightarrow{i + }u\cos v\vec{j}$$

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = -2u^2 \sin v \vec{j} + u\cos^2 v\vec{k} + u\sin^2 v\vec{k} - 2u^2 \cos v\,\vec{i}$$

$$= -2u^2 \cos v\,\vec{i} - 2u^2 \sin v\vec{j} + u\vec{k}$$

Note that for $v = \theta = 0, \qquad r = u = 1$, and we have

$$\vec{r_u} \times \vec{r_v} = -2\vec{i} + \vec{k}$$

which is inner.
Therefore we use

$$-\vec{r_u} \times \vec{r_v} = 2u^2 \cos v\,\vec{i} + 2u^2 \sin v\vec{j} - u\vec{k}$$

$$\vec{F} = (u\cos v - u\sin v + u^2)\vec{i} + 2u\cos v\vec{j} + \vec{k}$$

$\Rightarrow$

$$\vec{F} \cdot \vec{n} = 2u^3 \cos^2 v - 2u^3 \sin v\cos v + 2u^4 \cos v + 4u^3 \sin v\cos v - u$$

Therefore

$$\int\int_{S_2} \vec{F} \cdot \vec{n}ds = \int_0^{2\pi}\int_0^1 [2u^3 \cos^2 v + 2u^3 \sin v\cos v + 2u^4 \cos v - u]dudv$$

$$= \int_0^{2\pi} [\frac{1}{2}\cos^2 v + \frac{1}{2}\sin v\cos v + \frac{2}{5}\cos v - \frac{1}{2}]dv$$

$$= \int_0^{2\pi} \{\frac{1}{4}(1 + \cos 2v)\}dv + [\frac{1}{4}\sin^2 v + \frac{2}{5}\sin v - \frac{1}{2}v]_0^{2\pi}$$

$$= \left[\frac{v}{4} + \frac{\sin 2v}{8}\right]_0^{2\pi} - \pi$$

Thus

$$\int\int_{S_2} \vec{F} \cdot \vec{n}ds = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$

On S_1 : this is the circle $x^2 + y^2 \leq 1, \qquad z = 1$. We use the parametrization

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = 1$$

Therefore $\vec{r}(u,v) = u\cos v\vec{i} + u\sin v\vec{j} + \vec{k} \qquad 0 \leq u \leq 1, \qquad 0 \leq v \leq 2\pi$

$$\vec{r_u} = \cos\vec{v} + \sin v\vec{j} \qquad \vec{r_v} = -u\sin v\,\vec{i} + u\cos v\vec{j}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = u\cos^2 v\vec{k} + u\sin^2 v\vec{k} = u\vec{k}$$

As expected this is outward since $0 \le u \le 1$.
$\vec{F} = (u\cos v - u\sin v + 1)\vec{i} + 2u\cos v\vec{j} + \vec{k}$
$\Rightarrow$

$$\vec{F} \bullet (\vec{r}_u \times \vec{r}_v) = u$$

$\Rightarrow$

$$\int\int_{S_1} \vec{F} \bullet \vec{n} ds = \int_0^{2\pi}\int_0^1 u du dv = \pi$$

So that

$$\iint_{S_2} + \iint_{S_1} = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$$

Example Evaluate $\iint_S \vec{F} \bullet \vec{n} ds,$ where

$$\vec{F}(x,y,z) = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

and S is the positively oriented surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and $z = 0$ and $z = 2$ and $\vec{n}$.
Use the Divergence Theorem. Then

$$\iint_S \vec{F} \bullet \vec{n} ds = \iiint_V \operatorname{div} \vec{F} dV$$

$$div\vec{F} = 3\left(x^2 + y^2 + z^2\right)$$

$$\iiint_V div\vec{F} dv = \iiint_V 3\left(x^2 + y^2 + z^2\right) dv = 3\int_0^{2\pi}\int_0^1\int_0^2 \left(r^2 + z^2\right) r dz dr d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 \left(2r^3 + \frac{8}{3}r\right) dr d\theta = 3\int_0^{2\pi}\left(\frac{1}{2} + \frac{4}{3}\right) d\theta = 3(2\pi)\frac{11}{6} = 11\pi$$

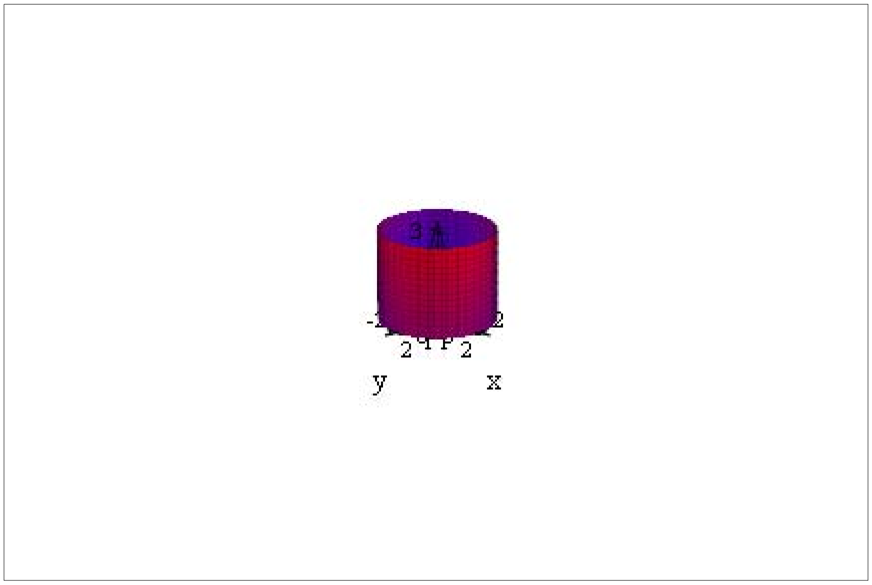
Example Let S be the closed surface of the solid cylinder T bounded by the planes $z = 0$ and $z = 3$ and the cylinder $x^2 + y^2 = 4.$ Calculate the surface integral

$$\iint_S \vec{F} \bullet \vec{n} dS$$

where

$$\vec{F} = \left(x^2 + y^2 + z^2\right)\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

Solution: The surface is
$x^2 + y^2 - 4 = 0$



We use the divergence Theorem

$$\iint_S \vec{F} \bullet n ds = \iiint_T \nabla \bullet \vec{F} dV$$

$$\vec{F} = \left(x^2 + y^2 + z^2\right)\left(x\vec{i}\right) + \left(x^2 + y^2 + z^2\right)\left(y\vec{j}\right) + \left(x^2 + y^2 + z^2\right)\left(z\vec{k}\right)$$

$$div\vec{F} = 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2 = 5\left(x^2 + y^2 + z^2\right)$$

Then

$$\iint_S \vec{F} \bullet n ds = \iiint_T 5\left(x^2 + y^2 + z^2\right) dV$$

Using cylindrical coordinates to evaluate the integral we have

$$\iint_S \vec{F} \bullet n ds = \iiint_T 5\left(x^2 + y^2 + z^2\right) dV = \int_0^{2\pi} \int_0^2 \int_0^3 \left(5\left(r^2 + z^2\right) r\right) dz dr d\theta = 300\pi$$

As an alternative, we can calculate the surface integral directly. Since the surface, *S*, is made up of three components, top (disc), side (cylinder) and bottom (disc), we deal with each components separately and then add the results.

$$\iint_S \vec{F} \bullet n ds = \iiint_T 5\left(x^2 + y^2 + z^2\right) dV = \int_0^{2\pi} \int_0^2 \int_0^3 \left(5\left(r^2 + z^2\right) r\right) dz dr d\theta = 300\pi$$

As an alternative, we can calculate the surface integral directly. Since the surface, *S*, is made up of three components, top (disc), side (cylinder) and bottom (disc), we deal with each components separately and then add the results.

(i) On the top, $z = 3$ and $\vec{n} = \vec{k}$. Thus $\vec{F} = \left(x^2 + y^2 + 9\right)\left(x\vec{i} + y\vec{j} + 3\vec{k}\right)$ and

$$\iint_{top} \vec{F} \bullet \vec{n} ds = \iint_{x^2+y^2 \leq 4} 3\left(x^2 + y^2 + 9\right) dA$$

$$= 3\int_0^{2\pi}\int_0^2 \left(r^2 + 9\right) r dr d\theta$$

$$= 3\int_0^{2\pi}\int_0^2 \left(r^3 + 9r\right) dr d\theta$$

$$= 132\pi.$$

(ii) On the bottom, $z = 0$ and $\vec{n} = -\vec{k}$. (Remember, we must use the outward normal.) Thus $\vec{F} = \left(x^2 + y^2\right)\left(x\vec{i} + y\vec{j} + 0\vec{k}\right)$ and

$$\vec{F} \bullet \vec{n} = 0.$$

Hence,

$$\iint_{bottom} \vec{F} \bullet n ds = 0.$$

(iii) On the side, we use cylindrical coordinates to parametrize with $r = 2$. So, we have

$$\vec{r} = 2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k}$$

$$\vec{r}_\theta = -2\sin\theta\vec{i} + 2\cos\theta\vec{j}$$

$$\vec{r}_z = \vec{k}$$

$$\vec{r}_\theta \times \vec{r}_z = -2\sin\theta\vec{i} \times \vec{k} + 2\cos\theta\vec{j} \times \vec{k}$$

$$= 2\sin\theta\vec{j} + 2\cos\theta\vec{i}$$

Before proceding, we check that we have the correct (outward) normal. This is OK, so we move to the integrand. On the surface, we have

$$\vec{F} = \left(4 + z^2\right)\left(2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k}\right)$$

$$\vec{F} \bullet (\vec{r}_\theta \times \vec{r}_z) = \left(4 + z^2\right)\left(4\cos^2\theta + 4\sin^2\theta\right)$$

$$= 4\left(4 + z^2\right)$$

Thus,

$$\iint_{side} \vec{F} \bullet n ds = \int_0^3\int_0^{2\pi} 4\left(4 + z^2\right) d\theta dz$$

$$= 168\pi$$

Finally,

$$\iint_S \vec{F} \bullet n ds = \iint_{top} \vec{F} \bullet n ds + \iint_{side} \vec{F} \bullet n ds + \iint_{bottom} \vec{F} \bullet n ds$$

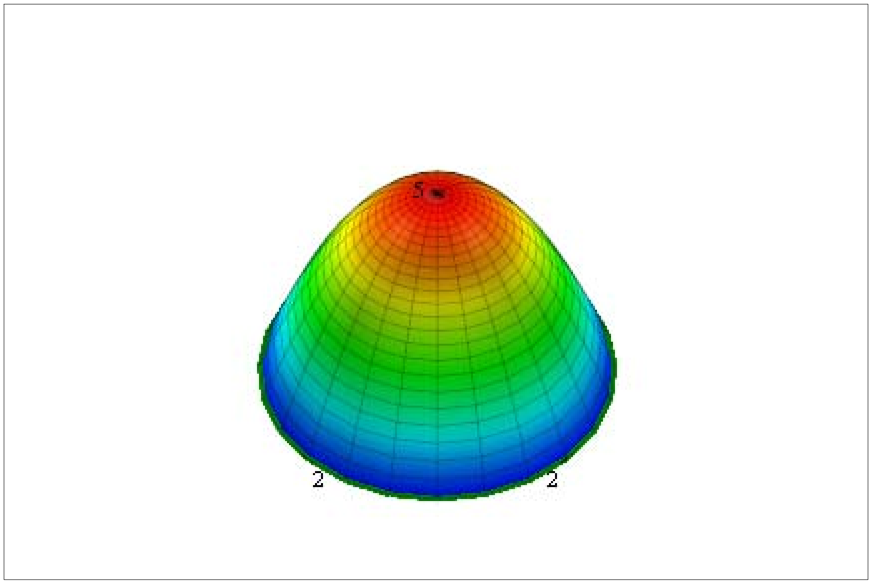
$$= 132\pi + 168\pi + 0 = 300\pi$$

Ma227 Additional Problems

Green's, Stokes' and the Divergence Theorems

Example Use Stokes' Theorem to evaluate $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}dS$ where $\vec{F} = z^2\vec{i} - 3x\vec{j} + x^3y^3\vec{k}$ and S is the part of the surface $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that S oriented upwards. Sketch S.

Solution: $(r, \theta, 5 - r^2)$



Stokes' Theorem is

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}dS = \oint_C \vec{F} \cdot d\vec{r}$$

Now the boundary C of S will be where the surface intersects $z = 1$, that is, when $1 = 5 - x^2 - y^2$ or $x^2 + y^2 = 4$. Thus

$$C: \; x = 2\cos t, y = 2\sin t; \;\; 0 \le t \le 2\pi; \; z = 1$$

and

$\vec{F} = z^2\vec{i} - 3x\vec{j} + x^3y^3\vec{k}$

$$\vec{r}(t) = 2\cos t\vec{i} + 2\sin t\vec{j} + \vec{k}$$
$$\vec{F}(t) = (1)^2\vec{i} - 3(2)\cos t\vec{j} + (2\cos t)^3(2\sin t)^3\vec{k}$$

Then

$$\vec{r}'(t) = -2\sin t\vec{i} + 2\cos t\vec{j}$$

and

$$\vec{F}(t) \cdot \vec{r}'(t) = -2\sin t - 12\cos^2 t$$

Thus

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(t)\cdot \vec{r}\,'(t)dt = \int_0^{2\pi}\left(-2\sin t - 12\cos^2 t\right)dt = [2\cos t - 6(\cos t\sin t + t)]_0^{2\pi} = -12\pi$$

Example Verify Green's theorem for the line integral

$$\oint_C (x+y)dx + (x-y)dy$$

where C is the positively oriented unit circle centered at the origin.
Solution: A parametrization of C is $x = \cos t, y = \sin t \;\; 0 \le t \le 2\pi.$ Thus

$$\oint_C (x+y)dx + (x-y)dy = \int_0^{2\pi} [(\cos t + \sin t)(-\sin t) + (\cos t - \sin t)(\cos t)]dt$$

$$= \int_0^{2\pi}\left(-2\sin t\cos t - \sin^2 t + \cos^2 t\right)dt$$

$$= \left[-2\sin^2 t + \frac{1}{2}\cos t\sin t - \frac{1}{2}t + \frac{1}{2}\cos t\sin t + \frac{1}{2}t\right]_0^{2\pi} = 0$$

Here $P = x+y$ and $Q = x-y$ so

$$\iint_{x^2+y^2\le 1} (Q_x - P_y)dA = \iint_{x^2+y^2\le 1} [1-1]dA = 0$$

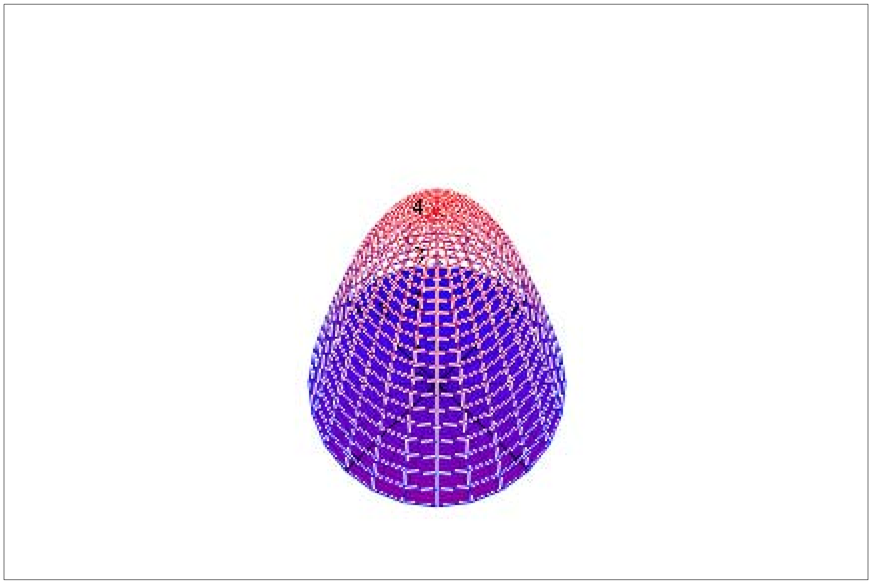
Example Evaluate the surface integral

$$\iint_S \vec{F}\cdot\vec{n}dS$$

where

$$\vec{F} = \left(xy\vec{i} - \frac{1}{2}y^2\vec{j} + z\vec{k}\right)$$

and the closed surface S consists of the two surfaces $z = 4 - 3x^2 - 3y^2,\; 0 \le z \le 4$ on the top on the top with normal upward, and $z = 0$ on the bottom with normal downward.
Solution: $\left(r,\theta,4-3r^2\right)$



We use the divergence theorem, namely

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_E \nabla \cdot \vec{F} dV$$

where E is the volume enclosed by S.

$$\nabla \cdot \vec{F} = y - y + 1 = 1$$

Note that $z = 0 \Rightarrow x^2 + y^2 = \frac{4}{3}$. Using cylindrical coordinates we have $0 \le z \le 4 - 3r^2$, $0 \le r \le \frac{2}{\sqrt{3}}$, and $0 \le \theta \le 2\pi$

$$\iiint_E \nabla \cdot \vec{F} dV = \int_0^{2\pi} \int_0^{\frac{2}{\sqrt{3}}} \int_0^{4-3r^2} (1) r dz dr d\theta = \int_0^{2\pi} \int_0^{\frac{2}{\sqrt{3}}} \left(4r - 3r^3\right) dr d\theta = \int_0^{2\pi} \left[2r^2 - \frac{3}{4}r^4\right]_0^{\frac{2}{\sqrt{3}}} d\theta$$

$$= \int_0^{2\pi} \left[2\left(\frac{4}{3}\right) - \left(\frac{3}{4}\right)\left(\frac{16}{9}\right)\right] d\theta = \int_0^{2\pi} \left(\frac{4}{3}\right) d\theta = \frac{8\pi}{3}$$

Alternatively, we will calculate the suface integral directly. Let S_1 denote the portion of the paraboloid on top and S_2 denote the disc on the bottom.

For S_1 we parametrize the surface using x and y as the parameters. Thus

$$\vec{r}(x,y) = \left\langle x, y, 4 - 3\left(x^2 + y^2\right)\right\rangle$$

$$\vec{r}_x(x,y) = \langle 1, 0, -6x\rangle$$

$$\vec{r}_y(x,y) = \langle 0, 1, -6y\rangle$$

The domain of $\vec{r}(x,y)$ is the disc $D = \left\{(x,y) | 0 \leqq x^2 + y^2 \leqq \frac{4}{3}\right\}$.

Then,

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -6x \\ 0 & 1 & -6y \end{vmatrix} = 6x\vec{i} + 6y\vec{j} + \vec{k}$$

We observe that the z component is positive, so we have the correct orientaion of the normal.

$$\begin{aligned}
\iint_{S_1} \vec{F} \cdot \vec{n} dS &= \iint_D \left\langle xy, -\frac{1}{2}y^2, 4 - 3\left(x^2 + y^2\right) \right\rangle \cdot \langle 6x, 6y, 1 \rangle dA_{xy} \\
&= \iint_D \left[6x^2y - 3y^3 + 4 - 3\left(x^2 + y^2\right) \right] dA_{xy} \\
&= \int_0^{2\pi} \int_0^{\frac{2}{\sqrt{3}}} \left[6r^3 \cos^2\theta \sin\theta - 3r^3 \sin^3\theta + 4 - 3r^2 \right] r dr d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{2}{\sqrt{3}}} \left[6r^4 \cos^2\theta \sin\theta - 3r^4 \sin^3\theta + 4r - 3r^3 \right] dr d\theta \\
& \int_0^{2\pi} \left[\frac{6}{5} r^5 \cos^2\theta \sin\theta - \frac{3}{4} r^5 \sin^3\theta + 2r^2 - \frac{3}{4} r^4 \right] \Big|_{r=0}^{r=\frac{2}{\sqrt{3}}} d\theta \\
&= \int_0^{2\pi} \left[\frac{6}{5} \left(\frac{2}{\sqrt{3}} \right)^5 \cos^2\theta \sin\theta - \frac{3}{4} \left(\frac{2}{\sqrt{3}} \right)^5 \sin^3\theta + 2\frac{4}{3} - \frac{3}{4}\frac{16}{9} \right] d\theta \\
&= \frac{6}{5} \left(\frac{2}{\sqrt{3}} \right)^5 \left[\frac{-\cos^3\theta}{3} \right]_{\theta=0}^{\theta=2\pi} - \frac{3}{4} \left(\frac{2}{\sqrt{3}} \right)^5 \left[\frac{-\sin^2\theta \cos\theta - 2\cos\theta}{3} \right]_{\theta=0}^{\theta=2\pi} + \frac{4}{3} 2\pi \\
&= \frac{8\pi}{3}
\end{aligned}$$

For S_2, we also parametrize using x and y, but now $z = 0$ so it's much simpler.

$$\begin{aligned}
\vec{r}(x,y) &= \langle x, y, 0 \rangle \\
\vec{r}_x(x,y) &= \langle 1, 0, 0 \rangle = \vec{i} \\
\vec{r}_y(x,y) &= \langle 0, 1, 0 \rangle = \vec{j}
\end{aligned}$$

The domain is the same disc,D, as for S_1.

$$\vec{r}_x \times \vec{r}_y = \vec{i} \times \vec{j} = \vec{k}$$

We observe that this vector points upward and we need the downward normal, so we use the negative and have

$$\begin{aligned}
\iint_{S2} \vec{F} \cdot \vec{n} dS &= \iint_D \left\langle xy, -\frac{1}{2}y^2, 0 \right\rangle \cdot \langle 0, 0, -1 \rangle dA_{xy} \\
&= 0
\end{aligned}$$

Finally

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_{S_1} \vec{F} \cdot \vec{n} dS + \iint_{S2} \vec{F} \cdot \vec{n} dS$$

$$= \frac{8\pi}{3} + 0$$