Eigenvalues and Eigenvectors

Vector Spaces

Definition. A vector space V (or linear space) is a collection of objects together with two operations vector addition (+) and scalar multiplication (•) which has the following properties:

(1) For all $\vec{u}, \vec{v} \in V$, there corresponds a unique vector $\vec{u} + \vec{v}$ in V (closure)

(2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutivity)

(3) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity)

(4) There exists a vector $\vec{0} \in V$ such that $\vec{u} + \vec{0} = \vec{u}$ for all $\vec{u} \in V$ (identity)

(5) For each $\vec{u} \in V$ there exists a unique vector $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$

(6) For every scalar c and for each vector $\vec{u} \in V$ there exists a unique vector $c \cdot \vec{u}$ in V.

(7) For all scalars c and d and all vectors $\vec{u}, \vec{v} \in V$

(i) $c(d \cdot \vec{u}) = (cd) \cdot \vec{u}$

(ii) $1 \cdot \vec{u} = \vec{u}$

(iii) $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$

(iv) $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{u}$

Example. Let V consist of vectors which are points in n dimensional Euclidian space, i.e. $\vec{u} = \left(u_1, u_2, . . ., u_n\right)$.

Define addition by

$$\vec{u} + \vec{v} = \left(u_1, \ldots, u_n\right) + \left(v_1, \ldots, v_n\right) = \left(u_1 + v_1, \ldots, u_n + v_n\right)$$

and scaler multiplication by

$$c \cdot \vec{u} = c\left(u_1, \ldots, u_n\right) = \left(cu_1, \ldots, cu_n\right)$$

.

This space is called V_n.

Remark. If we let $A = [a_{ij}]_{n \times n}$ be a square matrix of scalars and write vectors in V_n as

$$X = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

then the product $Y = AX$ is also a vector in V_n. The product $Y = AX$ is called a *linear transformation* of the vector X.

Very often in mathematics one wants to know which vectors, if any, are left unchanged in direction by the transformation. Two nonzero vectors have the same direction if and only if one is a nonzero scalar multiple of the other. Thus if AX is to have the same direction as X we want

$$AX = rX \qquad r \quad \text{some constant}$$

Thus we want to know which vectors X satisfy

$$AX = rX = rIX$$

or

$$(A - rI)X = 0$$

This last equation is equivalent to the system

$$\begin{aligned} (a_{11} - r)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - r)x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - r)x_n &= 0 \end{aligned} \tag{1}$$

By Cramer's rule if the determinant of the coefficients of the above system is *not* zero then the only solution to (1) is

$$x_1 = x_2 = \cdots = x_n = 0,$$

the trivial solution.

This implies $X = 0$. Since we want a nontrivial solution (nonzero vectors), we want

$$\det(A - rI) = 0$$

i.e.

$$\begin{vmatrix} a_{11}-r & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22}-r & & & \\ . & & & & \\ . & & & & \\ a_{n1} & . & . & . & a_{nn}-r \end{vmatrix} = 0$$

Now

$$\det(A - rI) = (-1)^n r^n + \cdots = \text{polynomial of degree } n \text{ in } r.$$

Thus if r is a root of

$$p(r) = \det(A - rI)$$

there will exist a solution $x_1, . . . , x_n$ of $(A - rI)X = 0$ such that not all the x_i's are zero. This $\Rightarrow$ that the vector

$$X = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

for this value of r and these x_i's satisfies $AX = rX$ and hence has the same direction under transformation.

The values of r such that $\det(A - rI) = 0$ are called *eigenvalues*. The vector X corresponding to an eigenvalue is called an *eigenvector* of the matrix A.

Example Find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$.

$$\det(A - rI) = \begin{vmatrix} 3-r & -4 \\ -4 & -3-r \end{vmatrix} = -(3-r)(+3+r) - 16 = -9 + r^2 - 16 = r^2 - 25$$

Therefore

$$p(r) = r^2 - 25$$

and $p(r) = 0 \Rightarrow r = \pm 5$. Thus the eigenvalues are ± 5.

The system

$$(A - rI)X = 0$$

is

$$
\begin{aligned}
(3-r)x_1 - 4x_2 &= 0 \\
-4x_1 + (-3-r)x_2 &= 0
\end{aligned}
$$

If $r = 5 \Rightarrow \left.\begin{aligned} -2x_1 - 4x_2 &= 0 \\ -4x_1 - 8x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 + 2x_2 = 0$ or $x_1 = -2x_2$

$\Rightarrow$ eigenvector $(-2t, t)$ or $X_1 = \begin{bmatrix} -2t \\ t \end{bmatrix}$ or just $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

If $r = -5 \Rightarrow \left.\begin{aligned} +8x_1 - 4x_2 &= 0 \\ -4x_1 + 2x_2 &= 0 \end{aligned}\right\} \Rightarrow 2x_1 = x_2$

$\Rightarrow$ eigenvector $(t, 2t)$ or $X_2 = \begin{bmatrix} t \\ 2t \end{bmatrix}$ or just $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Note that

$$
AX_2 = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix} = \begin{bmatrix} -5t \\ -10t \end{bmatrix} = -5\begin{bmatrix} t \\ 2t \end{bmatrix} = -5X_2
$$

Example Find the eigenvalues and eigenvectors of

$$
A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}
$$

Solution:

$$
\left|\begin{matrix} 2-r & 0 & 0 \\ 1 & -r & 2 \\ 0 & 0 & 3-r \end{matrix}\right| \begin{matrix} 2-r & 0 \\ 1 & -r \\ 0 & 0 \end{matrix} = -r(3-r)(2-r)
$$

Thus the eigenvalues are $r = 0, 2, 3$.
The system $(A - rI)X = 0$ becomes

$$
\begin{aligned}
(2-r)x_1 &= 0 \\
x_1 - rx_2 + 2x_3 &= 0 \\
(3-r)x_3 &= 0
\end{aligned}
$$

For the eigenvalue 0 we have $x_1 = x_3 = 0,$ which gives the eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. For the eigenvalue 3 we have $x_1 = 0, x_2 = \frac{2}{3}x_3$ which yields the eigenvector $\begin{bmatrix} 0 \\ 1 \\ \frac{3}{2} \end{bmatrix}$.

For the eigenvalue 2 we have $x_1 - 2x_2 + 2x_3 = 0, x_3 = 0$ which yields the eigenvector $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

You may view a slide show that illustrates how to find the eigenvalues and eigenvectors of a matrix by holding down the Ctrl key and clicking on Eigenvalues. You may download the tex file from which this slide show was made by holding down the Ctrl key and clicking on Eigenvalue Text.

Example Repeated Eigenvalues Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix}$$

Solution: $\det(A - rI) = \begin{vmatrix} 1-r & -2 & 4 \\ 3 & -4-r & 4 \\ 3 & -2 & 2-r \end{vmatrix} = -r^3 - r^2 + 8r + 12 = -(r-3)(r+2)^2$

Thus the eigenvalues are 3 and -2 and -2 is a repeated eigenvalue with multiplicity two. The system of equations $(A - rI)X = 0$ is, for this matrix,

$$\begin{aligned} (1-r)x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - (4+r)x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + (2-r)x_3 &= 0 \end{aligned}$$

Setting $r = 3$ yields

$$\begin{aligned} -2x_2 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 7x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 - x_3 &= 0 \end{aligned}$$

The augmented matrix for this system is $\begin{bmatrix} -2 & -2 & 4 & 0 \\ 3 & -7 & 4 & 0 \\ 3 & -2 & -1 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so the solutions of the above system are also the solutions of the system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Thus $x_1 = x_2 = x_3$ and an eigenvector corresponding to $r = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Setting $r = -2$ in the system $(A - rI)X = 0$ yields

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \end{aligned}$$

The augmented matrix for this system is $\begin{bmatrix} 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus, we have the one equation

$$x_1 - \frac{2}{3}x_2 + \frac{4}{3}x_3 = 0$$

To get two linearly independent vectors we first take $x_3 = 0$ and get $x_1 = \frac{2}{3}x_2$. Letting $x_2 = 1$ yields the eigenvector $\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$.

To get a second vector we set $x_2 = 0$ and get $x_1 = -\frac{4}{3}x_3$. Letting $x_3 = 1$ yields the eigenvector $\begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$.

We may check our results using SNB.

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix}, \text{ eigenvectors: } \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \right\} \leftrightarrow -2, \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 3$$

Example Complex Eigenvalues Find the eigenvalues and eigenvectors of the matrix A.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution. We note the following.

If $r_1 = \alpha + i\beta$ is a solution of the equation that determines the eigenvalues, namely,

$$p(r) = \det(A - rI) = 0$$

then $r_2 = \alpha - i\beta$ is also a solution of this equation, and hence is an eigenvalue. Recall that r_2 is called the complex conjugate of r_1 and $\bar{r}_1 = r_2$.

Let $\mathbf{z} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real vectors, be an eigenvector corresponding to r_1. Then it is not hard to see that $\bar{\mathbf{z}} = \mathbf{a} - i\mathbf{b}$ is an eigenvector corresponding to r_2. Since

$$A\mathbf{z} = r_1\mathbf{z} = r_1 I\mathbf{z}$$

then

$$(A - r_1 I)\mathbf{z} = 0$$

Taking the conjugate of this equation and noting that since A and I are real matrices then $\overline{A} = A$ and $\bar{I} = I$

$$\overline{(A - r_1 I)\mathbf{z}} = (A - \bar{r}_1 I)\bar{\mathbf{z}} = (A - r_2 I)\bar{\mathbf{z}} = 0$$

so $\bar{\mathbf{z}}$ is an eigenvector corresponding to r_2.

We find the eigenvalues for matrix A first.

$$\det(A - rI) = \left| \begin{array}{ccc|cc} 2-r & -1 & 0 & 2-r & -1 \\ 2 & 1-r & 1 & 2 & 1-r \\ 0 & 2 & 1-r & 0 & 2 \end{array} \right.$$

$$\begin{aligned} &= (2-r)(1-r)^2 - 2(1)(2-r) + 2(1-r) \\ &= (2-r)\left[\left(1 - 2r + r^2\right) - 2\right] + 2 - 2r \\ &= (2-r)\left(-1 - 2r + r^2\right) + 2 - 2r \\ &= -2 + r - 4r + 2r^2 + 2r^2 - r^3 + 2 - 2r \\ &= -5r + 4r^2 - r^3 = -r\left(r^2 - 4r + 5\right) \end{aligned}$$

Clearly one root is $r = 0$. Using the quadratic formula, the others are

$$\begin{aligned} r &= \frac{4 \pm \sqrt{4^2 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} \\ &= 2 \pm i \end{aligned}$$

The system of equations for the eigenvectors is

$$\begin{aligned} (2-r)x_1 - x_2 &= 0 \\ 2x_1 + (1-r)x_2 + x_3 &= 0 \\ 2x_2 + (1-r)x_3 &= 0 \end{aligned}$$

For $r = 0$, we solve

$$(A - 0I)X = 0$$

Using elimination on the augmented matrix, we have

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -.5 & 0 & 0 \\ 0 & 1 & .5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & .25 & 0 \\ 0 & 1 & .5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = -\frac{1}{4}x_3$ and $x_2 = -\frac{1}{2}x_3$ where x_3 is arbitrary. Letting $x_3 = 4$ we have that the eigenvector is any multiple of

$$\begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$$

Similarly, for $r = 2 + i$, we have the following. [The first step is an extra step of multiplying the first row by $2i$ to show how this goes.]

$$\begin{bmatrix} -i & -1 & 0 & 0 \\ 2 & -1-i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -2i & 0 & 0 \\ 2 & -1-i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -2i & 0 & 0 \\ 0 & -1+i & 1 & 0 \\ 0 & 2 & -1-i & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 2 & -2i & 0 & 0 \\ 0 & 2 & -1-i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1-i & 0 \\ 0 & 2 & -1-i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $2x_1 = (-1+i)x_3$ and $2x_2 = (1+i)x_3$. Again, the third component is arbitrary and any multiple of

$$\begin{bmatrix} -1+i \\ 1+i \\ 2 \end{bmatrix}$$

is an eigenvector.

Finally, since the entries in the matrix are all real, both eigenvalues and eigenvectors come in complex conjugate pairs and for $r = 2 - i$, eigenvectors are multiples of

$$\begin{bmatrix} -1-i \\ 1-i \\ 2 \end{bmatrix}.$$