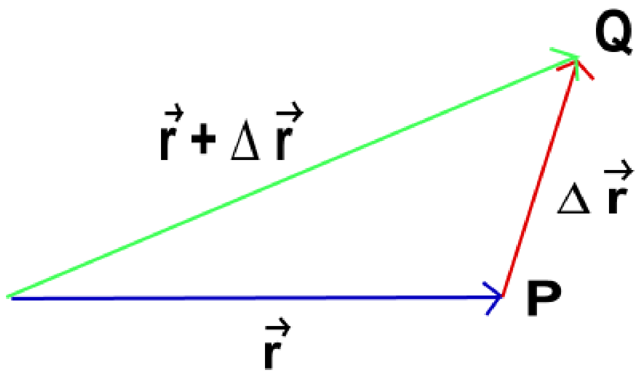
Ma 227

The Directional Derivative and the Gradient

Let $\Phi(x,y,z)$ be a scalar function with first partial derivatives $\Phi_x, \Phi_y,$ and Φ_z in some region of x,y,z –space. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the vector drawn from the origin to the point $P = (x,y,z)$. Suppose that we move from P to a nearby point $Q = (x + \Delta x, y + \Delta y, z + \Delta z)$.



Then Φ will change by an amount $\Delta\Phi$ where

$$\Delta\Phi = \Phi_x \Delta x + \Phi_y \Delta y + \Phi_z \Delta z + \epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z$$

where $\epsilon_1, \epsilon_2,$ and $\epsilon_3 \to 0$ as the point $Q \to P$. If we divide the change $\Delta\Phi$ by the distance $\Delta s = |\Delta\vec{r}|$ between P and Q, we obtain a measure of the rate at which Φ changes when we move from P to Q :

$$\frac{\Delta\Phi}{\Delta s} = \Phi_x \frac{\Delta x}{\Delta s} + \Phi_y \frac{\Delta y}{\Delta s} + \Phi_z \frac{\Delta z}{\Delta s} + \epsilon_1 \frac{\Delta x}{\Delta s} + \epsilon_2 \frac{\Delta y}{\Delta s} + \epsilon_3 \frac{\Delta z}{\Delta s}$$

Example:

If $\Phi(x,y,z)$ represents the temperature at any point $P(x,y,z)$ then $\frac{\Delta\Phi}{\Delta s}$ is the average rate of change in temperature per unit length at the point P in the direction in which Δs is measured.

The limiting value of $\frac{\Delta\Phi}{\Delta s}$ as $\Delta s \to 0$, that is, as $Q \to P$ along the segment PQ, is called the derivative of Φ in the direction PQ or simply the directional derivative of Φ. Since $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ as $Q \to P$, we have that

$$\frac{d\Phi}{ds} = \frac{\partial\Phi}{\partial x}\frac{dx}{ds} + \frac{\partial\Phi}{\partial y}\frac{dy}{ds} + \frac{\partial\Phi}{\partial z}\frac{dz}{ds}$$

The first factor in each term of the products in the expression above for the directional derivative depend only on Φ and the point P. The second factors in the products are independent of Φ and depend on the direction in which the derivative is being computed. We may rewrite the expression above in the form

$$\frac{d\Phi}{ds} = \left(\Phi_x\vec{i} + \Phi_y\vec{j} + \Phi_z\vec{k}\right) \cdot \left(\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} + \frac{dz}{ds}\vec{k}\right)$$
$$= \left(\Phi_x\vec{i} + \Phi_y\vec{j} + \Phi_z\vec{k}\right) \cdot \frac{d\vec{r}}{ds}$$

The vector $\Phi_x\vec{i} + \Phi_y\vec{j} + \Phi_z\vec{k}$ is known as the gradient of Φ or $grad\Phi$. Thus

$$grad\Phi = \Phi_x\vec{i} + \Phi_y\vec{j} + \Phi_z\vec{k}$$

The notation $\nabla\Phi$ is often used for $grad\Phi$. In this notation the operator ∇ is defined as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Example:

Let $\Phi(x,y,z) = xyz + 3x^4y^2z^3$. Then $\nabla\Phi = (yz + 12x^3y^2z^3)\vec{i} + (xz + 6x^4yz^3)\vec{j} + (xy + 9x^4y^2z^2)$

Example:

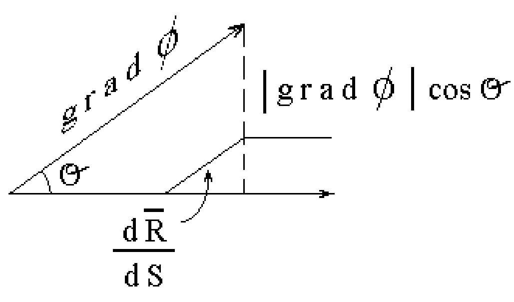
We may use SNB to find the gradient of a function. However, SNB writes vectors as ordered triples instead of the form given in the previous example. Thus

$\nabla(xyz + 3x^4y^2z^3) = (yz + 12x^3y^2z^3, xz + 6x^4yz^3, xy + 9x^4y^2z^2)$

With this notation we may write the directional derivative of Φ in the form

$$\frac{d\Phi}{ds} = grad\Phi \cdot \frac{d\vec{r}}{ds} = \nabla\Phi \cdot \frac{d\vec{r}}{ds}$$

Remark: Since Δs is the length of $\Delta\vec{r}$ then $\frac{\Delta\vec{r}}{\Delta s}$ and hence $\frac{d\vec{r}}{ds}$ are unit vectors. Therefore, $\nabla\Phi \cdot \frac{d\vec{r}}{ds}$ is the projection of $grad\Phi$ in the direction of $\frac{d\vec{r}}{ds}$. Thus $\nabla\Phi$ has the property that its projection in any direction is equal to the derivative of Φ in that direction. Since the maximum projection of a vector is the vector itself, it is clear that $grad\Phi$ extends in the direction of the greatest rate of change of Φ and has that rate of change for its length.



Example:

What is the directional derivative of the function $\Phi = xy^2 + yz^3$ at $(2,-1,1)$ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$?

$\nabla(xy^2 + yz^3) = (y^2, 2xy + z^3, 3yz^2)$ and a unit vector in the given direction is $\frac{1}{3}(1,2,2)$. Thus

$$\frac{d\Phi}{ds} = (y^2, 2xy + z^3, 3yz^2) \bullet \frac{1}{3}(1,2,2) = \tfrac{1}{3}y^2 + \tfrac{4}{3}xy + \tfrac{2}{3}z^3 + 2yz^2$$

Hence

$$\left.\frac{d\Phi}{ds}\right|_{(2,-1,1)} = \tfrac{1}{3}(-1)^2 + \tfrac{4}{3}(2)(-1) + \tfrac{2}{3}(1)^3 + 2(-1)(1)^2 = -\tfrac{11}{3}.$$

Remark There is a very nice discussion of the gradient at Gradient. There is a discussion of the gradient as well as a couple of very nice Java applets. This site was done at RPI.

Let us now consider the operator $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$. Given any other vector $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ where $\vec{F} = \vec{F}(x,y,z)$ we can consider $\nabla \bullet \vec{F}$, called the divergence of $\vec{F}$, and $\nabla \times \vec{F}$, called the curl $\vec{F}$

$$\nabla \bullet \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \bullet \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = div\vec{F}$$

$$\begin{aligned}
\nabla \times \vec{F} = curl\vec{F} &= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right) \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.
\end{aligned}$$

Example Let $\vec{F} = 2x\vec{i} + 3y^2\vec{j} + 2xz\vec{k}$. Find $\nabla \bullet \vec{F}$ and $\nabla \times \vec{F}$

$$div\vec{F} = 2 + 6y + 2z$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 3y^2 & 2xz \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k} - 0\vec{k} - 0\vec{i} - 2z\vec{j}$$