

Ma 227 Homework Solutions Fall 2013

Due 10/11/2013

13.4 p. 939 - 940 #1, 3, 7, 11, 13, 17

1) Evaluate the line integral by two methods: a) directly, and b) using Green's Theorem.

$$\oint_C (x - y)dx + (x + y)dy$$

C is the circle with center the origin and radius 2

a) Directly: Parametric equations for C are $x = 2 \cos t$, $y = 2 \sin t$, $0 \leq t \leq \pi$

$$\begin{aligned} & \oint_C (x - y)dx + (x + y)dy \\ &= \int_0^{2\pi} [(2 \cos t - 2 \sin t)(-2 \sin t) + (2 \cos t + 2 \sin t)(2 \cos t)]dt \\ &= \int_0^{2\pi} (4 \sin^2 t + 4 \cos^2 t)dt = \int_0^{2\pi} 4dt = 8\pi \end{aligned}$$

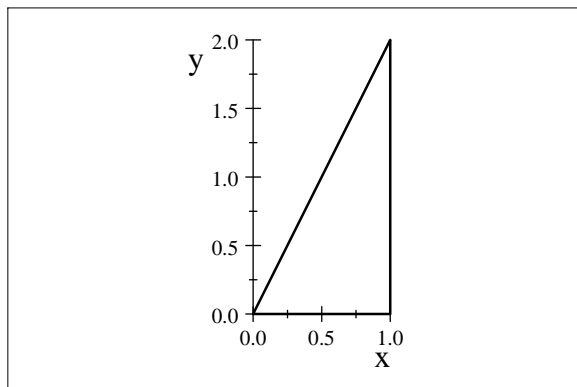
b) Using Greens Theorem:

$$\begin{aligned} \oint_C (x - y)dx + (x + y)dy &= \iint_D \left[\frac{\partial}{\partial x}(x + y) - \frac{\partial}{\partial y}(x - y) \right] dA \\ &= \iint_D [1 - (-1)]dA = 2 \iint_D dA = 2\pi(2)^2 = 8\pi \end{aligned}$$

3) Evaluate the line integral by two methods: a) directly, and b) using Green's Theorem.

$$\oint_C xydx + x^2y^3dy$$

(0,0,1,0,1,2,0,0)



a) directly: Let C_1 be the segment from (0,0) to (2,0), C_2 the segment from (2,0) to (2,3), C_3 the segment from (2,3) to (0,3), C_4 the segment from (0,3) to (0,0)

a) directly: Let C_1 be the segment from (0,0) to (1,0), C_2 the segment from (1,0) to (1,2)

and C_3 the segment from $(1, 2)$ to $(0, 0)$.

$$C_1 \rightarrow x = t, dx = dt; y = 0, dy = 0 \quad 0 \leq t \leq 1$$

$$C_2 \rightarrow x = 1, dx = 0; y = t, dy = dt \quad 0 \leq t \leq 2$$

$$C_3 \rightarrow \text{The equation of this line is } y = 2x. \text{ Thus } x = t, dx = dt; y = 2t, dy = 2dt \quad t : 1 \rightarrow 0$$

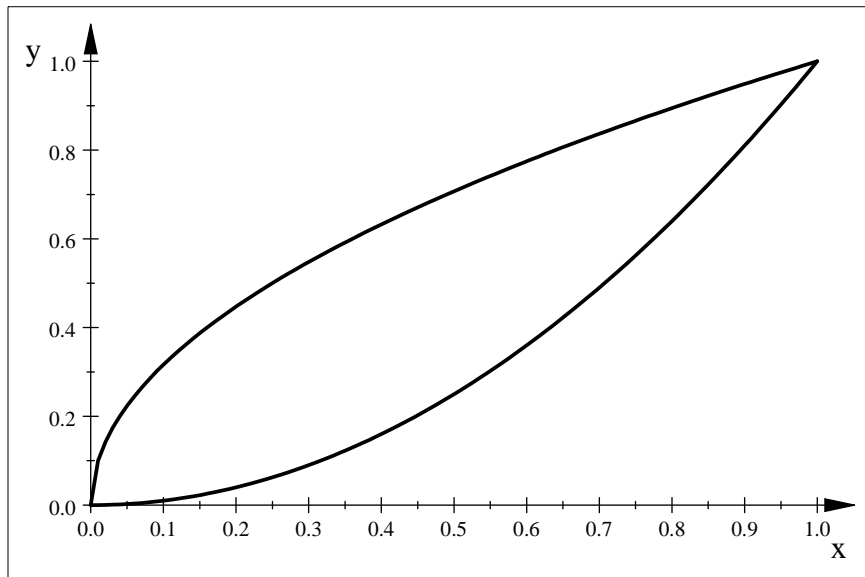
$$\begin{aligned} & \oint_C xydx + x^2y^3dy \\ &= \int_{C_1} xydx + x^2y^3dy + \int_{C_2} xydx + x^2y^3dy + \int_{C_3} xydx + x^2y^3dy \\ &= \int_0^1 0dt + \int_0^2 t^3dt + \int_1^0 t(2t)dt + \int_1^0 t^2(8t^3)(2dt) \\ &= \int_0^2 t^3dt + \int_1^0 (2t^2)dt + 2 \int_1^0 (8t^5)dt = \frac{2}{3} \end{aligned}$$

b) Using Greens Theorem:

$$\begin{aligned} \oint_C xydx + x^2y^3dy &= \iint_D [\partial/\partial x(x^2y^3) - \partial/\partial y(xy)]dA \\ &= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = 2/3 \end{aligned}$$

7) Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\oint_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy, C \text{ is the boundary of the region enclosed by the parabolas } y = x^2 \text{ and } x = y^2.$$



$$\oint_C (y + e^{\sqrt{x}}) dx + (2x + \cos y^2) dy = \iint_D \left[\frac{\partial}{\partial x} (2x + \cos y^2) - \frac{\partial}{\partial y} (y + e^{\sqrt{x}}) \right] dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2 - 1) dy dx = \int_0^1 (y^{\frac{1}{2}} - y^2) dy = \frac{1}{3}$$

11) $F(x, y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y} \rangle$,

C consists of the arc of the curve $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0, 0)$

C is traversed clockwise, so $-C$ gives the positive orientation

$$\int_C F \cdot dr = - \int_{-C} (\sqrt{x} + y^3) dx + (x^2 + \sqrt{y}) dy$$

$$= - \iint_D \left[\frac{\partial}{\partial x} (x^2 + \sqrt{y}) - \frac{\partial}{\partial y} (\sqrt{x} + y^3) \right] dA$$

$$= - \int_0^\pi \int_0^{\sin x} (2x - 3y^2) dy dx = - \int_0^\pi [2xy - y^3]_{y=0}^{y=\sin x} dx$$

$$= - \int_0^\pi (2x \sin x - \sin^3 x) dx = - \int_0^\pi (2x \sin x - (1 - \cos^2 x) \sin x) dx \quad [\text{Use integration by parts}]$$

$$= - \left[2 \sin x - 2x \cos x + \cos x - \frac{1}{3} \cos^3 x \right]_0^\pi = - \left(2\pi - 2 + \frac{2}{3} \right) = \frac{4}{3} - 2\pi$$

13) $F(x, y) = \langle e^x + x^2 y, e^y - xy^2 \rangle$,

C is the circle $x^2 + y^2 = 25$ orientated clockwise.

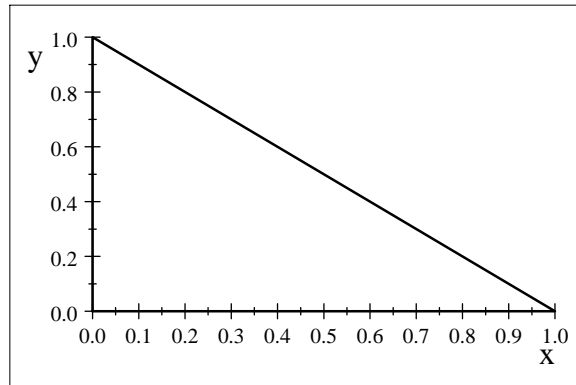
C is traversed clockwise, so $-C$ gives the positive orientation

$$\begin{aligned}
\int_C F \cdot dr &= -\int_{-C} (e^x + x^2y)dx + (e^y - xy^2)dy \\
&= -\iint_D \left[\frac{\partial}{\partial x}(e^y - xy^2) - \frac{\partial}{\partial y}(e^x + x^2y) \right] dA \\
&= -\iint_D (-y^2 - x^2)dA = \iint_D (x^2 + y^2)dA = \int_0^{2\pi} \int_0^5 (r^2) r dr d\theta \\
&= \int_0^{2\pi} d\theta \int_0^5 (r^3) dr = 2\pi \left[\frac{1}{4} r^4 \right]_0^5 = \frac{625}{2} \pi
\end{aligned}$$

17) Use Green's Theorem to find the work done by the force $F(x, y) = x(x + y)\vec{i} + xy^2\vec{j}$ in moving a particle from the origin along the x-axis to (1, 0) then along the line segment to (0, 1), and the back to the origin along the y-axis.

The path is shown below.

(0, 0, 1, 0, 0, 1, 0, 0)



The line joining (1, 0) to (0, 1) has equation $y = 1 - x$. Thus
By Greens Theorem,

$$Work = \int_C F \cdot dr = \oint_C x(x + y)dx + xy^2dy = \iint_D (y^2 - x)dydx$$

where C is the path described in the question and D is the triangle bounded by C .

$$Work = \int_0^1 \int_0^{1-x} (y^2 - x)dydx = -\frac{1}{12}$$

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5) Evaluate $\iint_S (x + y + z)dS$ $S : x = u + v, \quad y = u - v \quad z = 1 + 2u + v$

$$\iint_S (x + y + z)dS = \iint_D f(r(u, v))|r_u \times r_v|dA$$

define: $\vec{r}(u, v) = (u + v)\vec{i} + (u - v)\vec{j} + (1 + 2u + v)\vec{k}$ $0 \leq u \leq 2, 0 \leq v \leq 1$

$$\vec{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (1, 1, 2)$$

$$\vec{r}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (1, -1, 1)$$

$$\vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = 3\vec{i} + \vec{j} - 2\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$$

$$\begin{aligned} \iint_S (x + y + z) dS &= \iint_D f(r(u, v)) |\vec{r}_u \times \vec{r}_v| dA \\ &= \iint_D (u + v + u - v + 1 + 2u + v) |\vec{r}_u \times \vec{r}_v| dA \\ &= \int_0^1 \int_0^2 (4u + v + 1) \sqrt{14} \, du dv \\ &= \sqrt{14} \int_0^1 [2u^2 + uv + u]_{u=0}^{u=2} dv \\ &= \sqrt{14} \int_0^1 (2v + 10) dv = \sqrt{14} [v^2 + 10]_0^1 = 11\sqrt{14} \end{aligned}$$

7) $\iint_S y dS$, S is the helicoid with vector equation

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, 0 \leq u \leq 1, 0 \leq v \leq \pi$$

$$\vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{bmatrix} = \sin v \vec{i} - \cos v \vec{j} + u \vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$$

$$\begin{aligned} \iint_S y dS &= \iint_D (u \sin v) |\vec{r}_u \times \vec{r}_v| dA = \int_0^1 \int_0^\pi (u \sin v) \sqrt{1 + u^2} \, dv du \\ &= \int_0^1 u \sqrt{1 + u^2} \, du \int_0^\pi \sin v \, dv = \left[\frac{1}{3} (u^2 + 1)^{\frac{3}{2}} \right]_0^1 [-\cos v]_0^\pi \\ &= \frac{1}{3} [2^{\frac{3}{2}} - 1] (2) = \frac{2}{3} (2\sqrt{2} - 1) \end{aligned}$$