Ma 227 Homework 10 Solutions Fall 2011
Due 11/18/2011

13.4 p. 939 #1, 3, 7, 11, 13, 17

1) Evaluate the line integral by two methods: a)directly, and b) using Green's Theorem.

$\oint_C (x-y)dx + \ (x+y)dy$

C is the circle with center the origin and radius 2

a) Directly: Parametric equations for C are $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le \pi$

$$\oint_C (x-y)dx + \ (x+y)dy$$
$$= \int_0^{2\pi} [(2\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)(2\cos t)]dt$$
$$= \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)dt = \int_0^{2\pi} 4dt = 8\pi$$

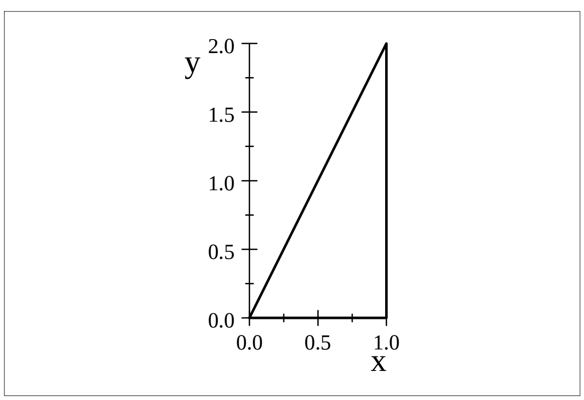
b) Using Greens Theorem:

$$\oint_C (x-y)dx + \ (x+y)dy = \int\int_D \left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y) \right] dA$$
$$= \int\int_D [1-(-1)]dA = 2\int\int_D dA = 2\pi(2)^2 = 8\pi$$

3) Evaluate the line integral by two methods: a)directly, and b) using Green's Theorem.

$\oint_C xydx + \ x^2y^3dy$

$(0,0,1,0,1,2,0,0)$



a) directly: Let C_1 be the segment from (0.0) to $(2,0)$, C_2 the segment from $(2,0)$ to $(2,3)$, C_3 the segment from $(2,3)$ to $(0,3)$, C_4 the segment from $(0,3)$ to $(0,0)$

a) directly: Let C_1 be the segment from (0.0) to $(1,0)$, C_2 the segment from $(1,0)$ to $(1,2)$

and C_3 the segment from $(1,2)$ to $(0,0)$.

$C_1 \to x = t, dx = dt; y = 0, dy = 0 \quad 0 \le t \le 1$

$C_2 \to x = 1, dx = 0dt; \quad y = t, dy = dt \quad 0 \le t \le 2$

$C_3 \to$The equation of this line is $y = 2x$. Thus $x = t, dx = dt; y = 2t, dy = 2dt \quad t : 1 \to 0$

$$\begin{aligned}
&\oint_C xydx + x^2y^3dy \\
&= \int_{C_1} xydx + x^2y^3dy + \int_{C_2} xydx + x^2y^3dy + \int_{C_3} xydx + x^2y^3dy \\
&= \int_0^1 0dt + \int_0^2 t^3dt + \int_1^0 t(2t)dt + \int_1^0 t^2(8t^3)(2dt) \\
&= \int_0^2 t^3dt + \int_1^0 (2t^2)dt + 2\int_1^0 (8t^5)dt = \frac{2}{3}
\end{aligned}$$

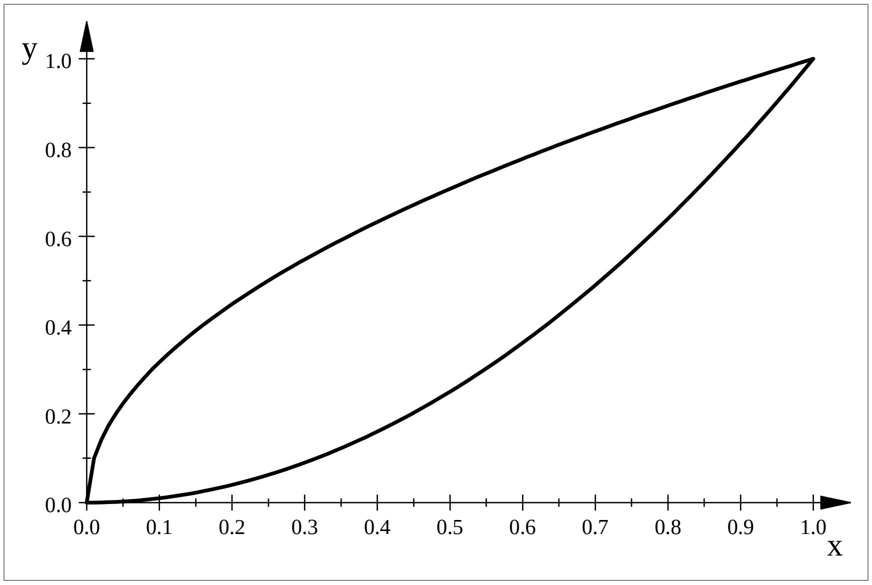
b) Using Greens Theorem:

$$\begin{aligned}
\oint_C xydx + x^2y^3dy &= \int\int_D [\partial/\partial x(x^2y^3) - \partial/\partial y(xy)]dA \\
&= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = 2/3
\end{aligned}$$

7) Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$\oint_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy$, C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

x^2



$$\oint_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy = \int\int_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}}) \right] dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2-1)dydx = \int_0^1 \left(y^{\frac{1}{2}} - y^2 \right) dy = \frac{1}{3}$$

11) $F(x,y) = \left\langle \sqrt{x} + y^3, x^2 + \sqrt{y} \right\rangle$,

C consists of the arc of the curve $y = \sin x$ from $(0,0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0,0)$

C is traversed clockwise, so $-C$ gives the positive orientation

$$\int_C F \cdot dr = -\int_{-C} (\sqrt{x} + y^3)dx + \left(x^2 + \sqrt{y}\right)dy$$

$$= -\int\int_D \left[\frac{\partial}{\partial x}\left(x^2 + \sqrt{y}\right) - \frac{\partial}{\partial y}(\sqrt{x} + y^3) \right] dA$$

$$= -\int_0^{\pi} \int_0^{\sin x} (2x - 3y^2)dydx = -\int_0^{\pi} [2xy - y^3]_{y=0}^{y=\sin x} dx$$

$$= -\int_0^{\pi} (2x\sin x - \sin^3 x)dx = -\int_0^{\pi} (2x\sin x - (1-\cos^2 x)\sin x)dx \qquad [\text{Use integration by parts}]$$

$$= -\left[2\sin x - 2x\cos x + \cos x - \frac{1}{3}\cos^3 x \right]_0^{\pi} = -\left(2\pi - 2 + \frac{2}{3}\right) = \frac{4}{3} - 2\pi$$

13) $F(x,y) = \langle e^x + x^2 y, e^y - xy^2 \rangle$,

C is the circle $x^2 + y^2 = 25$ orientated clockwise.

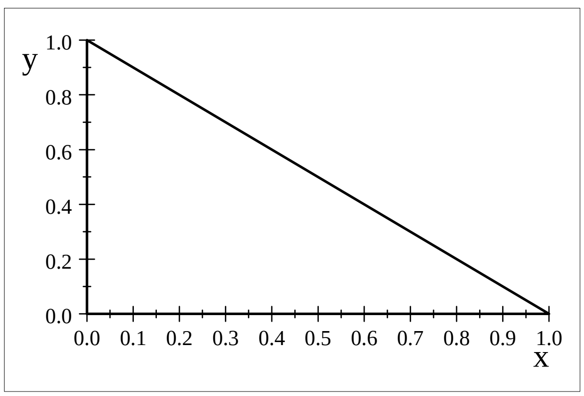
C is traversed clockwise, so $-C$ gives the positive orientation

$$\int_C F \cdot dr = -\int_{-C} (e^x + x^2y)dx + (e^y - xy^2)dy$$
$$= -\int\int_D \left[\frac{\partial}{\partial x}(e^y - xy^2) - \frac{\partial}{\partial y}(e^x + x^2y) \right] dA$$
$$= -\int\int_D (-y^2 - x^2)dA = \int\int_D (x^2 + y^2)dA = \int_0^{2\pi}\int_0^5 (r^2)rdrd\theta$$
$$= \int_0^{2\pi} d\theta \int_0^5 (r^3)dr = 2\pi\left[\frac{1}{4}r^4\right]_0^5 = \frac{625}{2}\pi$$

17) Use Green's Theorem to find the work done by the force $F(x,y) = x(x+y)\vec{i} + xy^2\vec{j}$ in moving a particle from the origin along the x-axis to $(1,0)$ then along the line segment to $(0,1)$, and the back to the origin along the y-axis.

The path is shown below.

$(0,0,1,0,0,1,0,0)$



The line joining $(1,0)$ to $(0,1)$ has equation $y = 1 - x$. Thus

By Greens Theorem,

$$Work = \int_C F.dr = \oint_C x(x+y)dx + xy^2dy = \int\int_D (y^2 - x)dydx$$

where C is the path described in the question and D is the triangle bounded by C.

Work $= \int_0^1\int_0^{1-x}(y^2 - x)dydx = -\frac{1}{12}$

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5) Evaluate $\int\int_s (x+y+z)dS \quad S : x = u+v, \quad y = u - v \quad z = 1 + 2u + v$

$$\int\int_s (x+y+z)dS = \int\int_D f(r(u,v))|r_u \times r_v|dA$$

define: $\vec{r}(u,v) = (u+v)\vec{i} + (u-v)\vec{j} + (1+2u+v)\vec{k} \quad 0 \le u \le 2, 0 \le v \le 1$

$\vec{r}_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) = (1,1,2)$

$\vec{r}_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) = (1,-1,1)$

$$\vec{r}_u \times \vec{r}_v = \det\begin{bmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = 3\vec{i} + \vec{j} - 2\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$$

$$\begin{aligned}\int\int_s (x+y+z)dS &= \int\int_D f(r(u,v))|r_u \times r_v|dA \\ &= \int\int_D (u+v+u-v+1+2u+v)|\vec{r}_u \times \vec{r}_v|dA \\ &= \int_0^1\int_0^2 (4u+v+1)\sqrt{14}\,dudv \\ &= \sqrt{14}\int_0^1 [2u^2 + uv + u]_{u=0}^{u=2} dv \\ &= \sqrt{14}\int_0^1 (2v+10)dv = \sqrt{14}\,[v^2+10]_0^1 = 11\sqrt{14}\end{aligned}$$

7) $\iint_S ydS$, S is the helicoid with vector equation

$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle, 0 \le u \le 1, 0 \le v \le \pi$

$$\vec{r}_u \times \vec{r}_v = \det\begin{bmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{bmatrix} = \sin v\vec{i} - \cos v\vec{j} + u\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

$$\begin{aligned}\iint_S ydS &= \iint_D (u\sin v)|\vec{r}_u \times \vec{r}_v|dA = \int_0^1\int_0^\pi (u\sin v)\sqrt{1+u^2}\,dvdu \\ &= \int_0^1 u\sqrt{1+u^2}\,du\int_0^\pi \sin v du = \left[\frac{1}{3}(u^2+1)^{\frac{3}{2}}\right]_0^1 [-\cos v]_0^\pi \\ &= \frac{1}{3}\left[2^{\frac{3}{2}} - 1\right]\end{aligned}$$

9) $\int\int_s x^2yzdS \qquad S$ is the part of the plane $z = 1+2x+3y$ that lies above $[0,3]\times[0,2]$

Then: $0 \le x \le 3 \qquad 0 \le y \le 2$

$\frac{\partial z}{\partial x} = 2 \qquad \frac{\partial z}{\partial y} = 3$

$$\int\int_s x^2yzdS = \int\int_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA$$
$$\int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{4+9+1}\,dydx$$
$$\int_0^3\int_0^2 \sqrt{14}(x^2y+2x^3y+3x^2y^2)dydx = 171\sqrt{14}$$

9) $\int\int_s yzdS$ S is the part of the plane $x+y+z=1$ that lies in the first octant

$z = 1-(x+y)$ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -1$

$$\int\int_D y(1-x-y)\sqrt{(-1)^2+(-1)^2+1^2}\,dA$$
$$= \int_0^1\int_0^{1-x}(\sqrt{3}y(1-x-y))dydx = \frac{1}{24}\sqrt{3}$$

17) $\int\int_s (x^2z+y^2z)dS$

S is the hemisphere $x^2+y^2+z^2 = 4,\ z \geq 0$

Using spherical coordinates :

$\vec{r}(\phi,\theta) = 2\sin\phi\cos\theta\vec{i} + 2\sin\phi\sin\theta\vec{j} + 2\cos\phi\vec{k}$

$|\vec{r}_\theta \times \vec{r}_z| = 4\sin\phi$

$$\int\int_s (x^2z+y^2z)dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)d\phi d\theta$$
$$= [16\pi\sin^4\phi]_0^{\frac{\pi}{2}} = 16\pi$$