Ma 227 Homework Fall 2013 Due 10/18/13

Section 13.6
959 #9, 11, 17, 21, 23, 25, 31

9) $\int\int_s x^2yzdS$ S is the part of the plane $z = 1 + 2x + 3y$ that lies above $[0,3] \times [0,2]$
Then: $0 \le x \le 3 \quad 0 \le y \le 2$
$\frac{\partial z}{\partial x} = 2 \quad \frac{\partial z}{\partial y} = 3$
$\int\int_s x^2yzdS = \int\int_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$
$\int_0^3\int_0^2 x^2y(1 + 2x + 3y)\sqrt{4 + 9 + 1}\, dydx$
$\int_0^3\int_0^2 \sqrt{14}\,(x^2y + 2x^3y + 3x^2y^2)dydx = 171\sqrt{14}$

11) $\int\int_s yzdS$ S is the part of the plane $x + y + z = 1$ that lies in the first octant
$z = 1 - (x + y) \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -1$

$$\int\int_D y(1 - x - y)\sqrt{(-1)^2 + (-1)^2 + 1^2}\, dA$$
$$= \int_0^1\int_0^{1-x}(\sqrt{3}\,y(1 - x - y))dydx = \frac{1}{24}\sqrt{3}$$

17) $\int\int_s (x^2z + y^2z)dS$
S is the hemisphere $x^2 + y^2 + z^2 = 4,\ z \ge 0$

Using spherical coordinates :
$\vec{r}(\phi,\theta) = 2\sin\phi\cos\theta\vec{i} + 2\sin\phi\sin\theta\vec{j} + 2\cos\phi\vec{k}$
$|\vec{r}_\theta \times \vec{r}_z| = 4\sin\phi$

$$\int\int_s (x^2z + y^2z)dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)d\phi d\theta$$
$$= [16\pi\sin^4\phi]_0^{\frac{\pi}{2}} = 16\pi$$

Evaluate $\int\int_s = \vec{F}\cdot d\vec{S}$ for the given vector field $\vec{F}$ and the oriented surface S.

21. $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$
S is the part of the parboloid $z = 4 - x^2 - y^2$ that lies above
$0 \le x \le 1, 0 \le y \le 1$ and has upward direction
$S : \vec{r}(u,v) = \vec{r}(x,y) = (x,y,4 - x^2 - y^2)$
$\vec{r}_x = (1,0,-2x)$
$\vec{r}_y = (0,1,-2y)$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = (2x, 2y, 1)$$

$\int\int_s = \vec{F} \cdot d\vec{S} = \int\int_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) dA$
$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = (xy, yz, zx) \cdot (2x, 2y, 1) = xz + 2x^2y + 2y^2z$
$\int_0^1 \int_0^1 (xz + 2x^2y + 2y^2z) dxdy$
$\int_0^1 \int_0^1 (x(4 - x^2 - y^2) + 2x^2y + 2y^2(4 - x^2 - y^2)) dxdy = \frac{713}{180}$

23. $\vec{F}(x,y,z) = xze^y \vec{i} - xze^y \vec{j} + z\vec{k}$
S is the part of the plane $x + y + z = 1$ in the first octant in the downward direction.
$\int\int_s = \vec{F} \cdot d\vec{S} \qquad \vec{r} = (x, y, 1 - x - y)$
$\vec{r}_x = (1, 0, -1)$
$\vec{r}_y = (0, 1, -1)$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1)$$

$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = (xze^y, -xze^y, z) \cdot (1, 1, 1) = z$
$\int\int_s \vec{F} \cdot d\vec{S} = \int\int_D z dA = -\int_0^1 \int_0^{1-x} (1 - x - y) dydx = -\frac{1}{6}$
(negative sign to indicate downward direction)

25. $\vec{F}(x,y,z) = x\vec{i} - z\vec{j} + y\vec{k}$
S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation toward the origin.

$$\int\int_s \vec{F} \cdot d\vec{S} = -\int\int_D \left[x(\tfrac{1}{2})(4 - x^2 - y^2)^{-\frac{1}{2}}(-2x) - (-z)\tfrac{1}{2}(4 - x^2 - y^2)^{-\frac{1}{2}}(-2y) + y \right] dA$$

$$= -\int\int_D \left(\frac{x^2}{\sqrt{4 - x^2 - y^2}} - \sqrt{4 - x^2 - y^2} \cdot \frac{y}{\sqrt{4 - x^2 - y^2}} + y \right) dA$$
$$= -\int\int_D x^2(4 - (x^2 + y^2))^{-\frac{1}{2}} dA$$

$$= -\int_0^{\frac{\pi}{2}} \int_0^2 (r\cos\theta)^2 (4 - r^2)^{-\frac{1}{2}} r dr d\theta$$
$$= -\int_0^{\frac{\pi}{2}} r\cos\theta d\theta \int_0^2 r^3 (4 - r^2)^{-\frac{1}{2}} dr \qquad [\text{Let } u = 4 - r^2 \Rightarrow r^2 = 4 - u \,\&\, -\tfrac{1}{2} du = r dr]$$

$$= -\int_0^{\frac{\pi}{2}} (\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta) d\theta \int_4^0 -\tfrac{1}{2}(4 - u)(u)^{-\frac{1}{2}} du$$
$$= -[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta]_0^{\frac{\pi}{2}} (-\tfrac{1}{2}) \left[8\sqrt{u} - \tfrac{2}{3}u^{\frac{3}{2}} \right]_4^0 = -\tfrac{\pi}{4}(-\tfrac{1}{2})\left(-16 + \tfrac{16}{3}\right) = -\tfrac{4}{3}\pi$$

31. $\vec{F}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

S is the boundary of the solid half cylinder $0 \leq z \leq \sqrt{1-y^2}$,$0 \leq x \leq 2$

Here S consists of four surfaces: S_1,the top surface (a portion of the circular cylinder $(y^2+z^2 = 1)$,S_2,the bottom surface (a portion of the $xy-plane$); S_3,the front half disk in the plane$x = 2$,and S_4,the back half disk in the plane $x = 0$

On S_1 the surface is $z = \sqrt{1-y^2}$ for $0 \leq x \leq 2, -1 \leq y \leq 1$ with upward direction:

$$\int\int_{s_1} \vec{F} \cdot d\vec{S} = \int_0^2 \int_{-1}^1 \left[-x^2(0) - y^2\left(-\frac{y}{\sqrt{1-y^2}}\right) + z^2 \right] dydx$$
$$= \int_0^2 \int_{-1}^1 \left[\left(-\frac{y^3}{\sqrt{1-y^2}} + 1 - y^2\right) \right] dydx$$

$$= \int_0^2 \left[\sqrt{1-y^2} + \tfrac{1}{3}(1-y^2)^{\frac{3}{2}} + y - \tfrac{1}{3}y^3 \right]_{y=-1}^{y=1} dx$$
$$= \int_0^2 \tfrac{4}{3} dx = \tfrac{8}{3}$$

On S_2 the surface is $z = 0$ with downward orientation:

$$\int\int_{s_1} \vec{F} \cdot d\vec{S} = \int_0^2 \int_{-1}^1 -z^2 dydx = \int_0^2 \int_{-1}^1 (0) dydx = 0$$

On S_3 the surface is $x = 2$ for $-1 \leq y \leq 1, 0 \leq z \leq \sqrt{1-y^2}$, oriented in the positive x-direction. Regarding y and z as parameters, we have:
$r_y \times r_z = \vec{i}$ and

$$\int\int_{s_3} \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2 dzdy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} 4dzdy = 4A(S_3) = 2\pi$$

On S_3 the surface is $x = 0$ for $-1 \leq y \leq 1, 0 \leq z \leq \sqrt{1-y^2}$, oriented in the negative x-direction. Regarding y and z as parameters, we have:
$-(r_y \times r_z) = \vec{i}$ and

$$\int\int_{s_3} \vec{F} \cdot d\vec{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2 dzdy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} 0dzdy = 0$$

Thus:

$$\int\int_{s} \vec{F} \cdot d\vec{S} = \tfrac{8}{3} + 0 + 2\pi + 0 = 2\pi + \tfrac{8}{3}$$