Ma 227 Homework Solutions Fall 2013 Due 10/25/2013

Page 965 # 3, 5, 9, Stokes' Theorem
Section 13.7
3. $\vec{F}(x,y,z) = x^2z^2\vec{i} + y^2z^2\vec{j} + xyz\vec{k}$, S is the par of the paraboloid $z = x^2 + y^2$ the lies inside the cylinder $x^2 + y^2 = 4$ oriented upward.

The parabloid intersects the cylinder in the circle $x^2 + y^2 = 4,\ z = 4.$ The boundary curve C should be oriented in the counterclockwise direction when viewed from above, so a vector equation of C is

$$\vec{r}(t) = 2\cos t\vec{i} + 2\sin t\vec{j} + 4\vec{k},\ \ 0 \le t \le 2\pi$$

Hence $\vec{r}'(t) = -2\sin t\vec{i} + 2\cos t\vec{j}$

$$F(\vec{r}(t)) = \left(4\cos^2 t\right)(16)\vec{i} + \left(4\sin^2 t\right)(16)\vec{j} + (2\cos t)(2\sin t)(4)\vec{k}$$

By Stokes' Theorem

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(r(t)) \cdot \vec{r}'(t)dt$$

$$= \int_0^{2\pi} \left(-18\cos^2 t\sin t + 128\sin^2 t\cos t + 0\right)dt$$

$$= 128\left[\frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\right]_0^{2\pi} = 0$$

5. C is the square in the plane $z = -1.$ By (3)

$$\iint_{S_1} \operatorname{curl}\vec{F}d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \operatorname{curl}\vec{F}d\vec{S}$$

where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube.

$$\operatorname{curl}\vec{F} = (x^2z)\vec{i} + (xy - 2xyz)\vec{j} + (y - xz)\vec{k}$$

For S_2 we choose $\vec{n} = \vec{k}$ so that C has the same orientation for both surfaces. Then

$$\operatorname{curl}\vec{F} \cdot \vec{n} = y - xz = x + y$$

since $z = -1.$ Thus

$$\iint_{S_2} \operatorname{curl}\vec{F} \cdot d\vec{S} = \int_{-1}^{1}\int_{-1}^{1}(x + y)dxdy = 0$$

so

$$\iint_{S_1} \operatorname{curl}\vec{F} \cdot d\vec{S} = 0$$

9.

$$\text{curl}\vec{F} = (xe^{xy} - y)\vec{i} - (ye^{xy} - y)\vec{j} - (2z - z)\vec{k}$$

Take the surface S to be the disk $x^2 + y^2 \le 16, z = 5$. Since C is oriented clockwise (from above), we orient S upward. Then $n = k$ and $curlF \cdot n = 2z - z$ on S were $z = 5.$ Thus

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_S \text{curl}\vec{F} \cdot nd\vec{S} = \iint_D [2z - z]dS$$

$$\iint_D (10 - 5)dS$$

$$= 5(Area)_S = 5\left(\pi \cdot 4^2\right) = 80\pi$$