Ma 227 Homework Solutions Due 11/15/2013

9.1 5, 7

5. Express the given d.e. as a matrix system in normal form:

$$x' = (\sin t)x + e^t y$$
$$y' = (\cos t)x + (a + bt^3)y$$

We start by expressing right-hand sides of both equations as dot products:

$$(\sin t)x + e^t y = [\sin t, e^t] \bullet [x, y], \qquad (\cos t)x + (a + bt^3)y = [\cos t] \bullet \left[a + bt^3\right]$$

Thus, by definition of the product of a matrix and column vector, the matrix form is guven by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \sin t & e^t \\ \cos t & a + bt^3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

7. Express the given higher-order d.e. as a matrix system in normal form:

$$my'' + by' + ky = 0$$

Let $x_1 = y$

$x_1' = y' = x_2$

$x_2' = y'' = \frac{-by'}{m} - \frac{ky}{m}$

$\Rightarrow \quad x_2' = -\frac{bx_2}{m} - \frac{kx_1}{m}$ so system is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

9.4 - 1, 7, 11, 17, 23

1. Write the given system in the matrix form $x' = Ax + f$

$$x'(t) = 3x(t) - y(t) + t^2$$
$$y'(t) = -x(t) + 2y(t) + e^t$$

Writing the matrix with the coefficients of x and y:

$$\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

Thus, this system becomes the equation in matrix form given by:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} +. \begin{bmatrix} t^2 \\ e^t \end{bmatrix}.$$

7. Rewrite the given scalar equation as a first order system in normal form. Express the system in the in the matrix form $x' = Ax + f$.

$$\frac{d^4w}{dt^4} + w = t^2.$$

Let $x_1(t) = w(t)$

$$x_1'(t) = \frac{dw}{dt} = x_2(t)$$
$$x_2'(t) = \frac{d^2w}{dt^2} = x_3(t)$$
$$x_3'(t) = \frac{d^3w}{dt^3} = x_4(t)$$
$$x_4'(t) = \frac{d^4w}{dt^4} = t^2 - w = t^2 - x_1(t)$$

So system is:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}$$

11. Write the given system as a set of scalar equations:

$$x' = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 5 \\ 0 & 5 & 1 \end{bmatrix} x + e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Multiplying the matrix values with the appropriate variable:

$$x_1' = 1(x_1)(t) + 0(x_2)(t) + 1(x_3)(t) + 1(e^t) + 0(t)$$
$$x_2' = -1(x_1)(t) + 2(x_2)(t) + 5(x_3)(t) + 0(e^t) + 1(t)$$
$$x_3' = 0(x_1)(t) + 5(x_2)(t) + 1(x_3)(t) + 0(e^t) + 0(t)$$

Simplifying:

$$x_1' = x_1(t) + x_3(t) + e^t$$
$$x_2' = -x_1(t) + 2x_2(t) + 5x_3(t) + t$$
$$x_3' = 5x_2(t) + x_3(t)$$

17. Determine whether the given vectors are linearly independent (LI) or linearly dependent (LD) on the interval $(-\infty, \infty)$

$$e^{2t}\begin{bmatrix}1\\0\\5\end{bmatrix}, e^{2t}\begin{bmatrix}1\\1\\-1\end{bmatrix}, e^{3t}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

By scalar multiplication:

$$\begin{bmatrix}e^{2t}\\0\\5e^{2t}\end{bmatrix}, \begin{bmatrix}e^{2t}\\e^{2t}\\-e^{2t}\end{bmatrix}, \begin{bmatrix}0\\e^{3t}\\0\end{bmatrix}$$

To be linearly independent:

$$c_1\begin{bmatrix}e^{2t}\\0\\5e^{2t}\end{bmatrix} + c_2\begin{bmatrix}e^{2t}\\e^{2t}\\-e^{2t}\end{bmatrix} + c_3\begin{bmatrix}0\\e^{3t}\\0\end{bmatrix} = 0$$

Writing as a system of equations:

$c_1 + c_2 = 0$
$c_2 + c_3 = 0$
$5c_1 - c_2 = 0$
Solving$\Rightarrow$ $c_1 = -c_2; c_2 = 0; c_3 = -c_2$
Therefore: $c_1 = c_2 = c_3 = 0$
This tells us that the original set of vectors mustbe LI on the interval $(-\infty, \infty)$

23. The vectors $\vec{x}_1 = \begin{bmatrix}e^{-t}\\2e^{-t}\\e^{-t}\end{bmatrix}, \vec{x}_2 = \begin{bmatrix}e^{t}\\0\\e^{t}\end{bmatrix}, \vec{x}_3 = \begin{bmatrix}e^{3t}\\-e^{3t}\\2e^{3t}\end{bmatrix}$ are solutions to the system $\vec{x}'(t) = A\vec{x}(t)$. Do they form a fundamental solution set?

Take wronskian: $\begin{bmatrix}e^{-t} & e^{t} & e^{3t}\\2e^{-t} & 0 & -e^{3t}\\e^{-t} & e^{t} & 2e^{3t}\end{bmatrix}$, determinant: $-2e^{3t} \neq 0$ then the system is linearly independant thus,

Fundamental Solution: $X(t) = \begin{bmatrix}e^{-t} & e^{t} & e^{3t}\\2e^{-t} & 0 & -e^{3t}\\e^{-t} & e^{t} & 2e^{3t}\end{bmatrix}$

General Solution: $X(t)c = c_1\begin{bmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{bmatrix} + c_2\begin{bmatrix} e^{t} \\ 0 \\ e^{t} \end{bmatrix} + c_3\begin{bmatrix} e^{3t} \\ -e^{3t} \\ 2e^{3t} \end{bmatrix}$

9.5 - 11, 13, 19, 21, [**32, 33 not assigned**]

11. Find a general solution to the system

$$A = \begin{bmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{bmatrix}$$

$$|A - rI| = \begin{bmatrix} -1-r & \frac{3}{4} \\ -5 & 3-r \end{bmatrix} = 0$$

$$r^2 - 2r + \frac{3}{4} = 0 \Rightarrow r = \frac{1}{2}, \frac{3}{2} \Rightarrow \text{eigenvalues}$$

$$\left(A - \frac{1}{2}r\right)u = \begin{bmatrix} -\frac{3}{2} & \frac{3}{4} \\ -5 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\frac{3}{2}u_1 + \frac{3}{4}u_2 = 0$$

$$\Rightarrow u_1 = \frac{1}{2}u_2 \Rightarrow u_2 = s; u_1 = \frac{1}{2}s$$

$$-5u_1 + \frac{5}{2}u_2 = 0$$

$$u = s\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow \textit{eigenvector}$$

$$\left(A - \frac{3}{2}r\right)u = \begin{bmatrix} -\frac{5}{2} & \frac{3}{4} \\ -5 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\frac{5}{2}u_1 + \frac{3}{4}u_2 = 0$$

$$\Rightarrow u_1 = \frac{6}{20}u_2 \Rightarrow u_2 = s; u_1 = \frac{3}{10}s$$

$$-5u_1 + \frac{3}{2}u_2 = 0$$

$$u = s\begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix} \Rightarrow \text{eigenvector}$$

General Solution: $X(t)c = c_1 e^{\frac{1}{2}t}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{\frac{3}{2}t}\begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix}$

13. Find a general solution to the system

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} 1-r & 2 & 2 \\ 2 & -r & 3 \\ 2 & 3 & -r \end{vmatrix} = 0$$

$$\Rightarrow (1-r)\begin{vmatrix} -r & 3 \\ 3 & -r \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 2 & -r \end{vmatrix} + 2\begin{vmatrix} 2 & -r \\ 2 & 3 \end{vmatrix} = 0$$

$\Rightarrow (1-r)(r^2 - 9) - 2(-2r - 6) + 2(6 + 2r) = (1-r)(r-2)(r+2) = 0$
$\Rightarrow (r+3)[(1-r)(r-3) + 8] = 0 \Rightarrow (r+3)(r-5)(r+1) = 0$
The eigenvalues are:
$r = -3, 5, -1$

Eigenvector for:

$r = -3$

$$(A + 3I)u = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4u_1 + 2u_2 + 2u_3 = 0$
$2u_1 + 3u_2 + 3u_3 = 0$
$2u_1 + 3u_2 + 3u_3 = 0$
Solving the system $\Rightarrow u_1 = 0; u_2 = -u_3;$
Setting $u_3 = s \Rightarrow u_2 = -s$

$$\Rightarrow u_1 = \begin{bmatrix} 0 \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$r = 5$

$$(A+3I)u = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4u_1 + 2u_2 + 2u_3 = 0$
$2u_1 - 5u_2 + 3u_3 = 0$
$2u_1 + 3u_2 - 5u_3 = 0$
Solving the system$\Rightarrow u_1 = u_3, u_2 = u_3$
Letting $u_3 = s \Rightarrow u_1 = u_2 = u_3 = s$

$$\Rightarrow u_2 = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$r = -1$

$$(A+3I)u = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2u_1 + 2u_2 + 2u_3 = 0$
$2u_1 + u_2 + 3u_3 = 0$
$2u_1 + 3u_2 + 1u_3 = 0$
Solving the system$\Rightarrow u_1 = -2u_2, u_3 = u_2$
Letting $u_2 = s \Rightarrow u_1 = -2s, u_3 = s$

$$\Rightarrow u_3 = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the general solution for the system is:

$$x(t) = c_1 e^{-3t} u_1 + c_2 e^{-t} u_2 + c_3 e^{5t} u_3 = c_1 e^{-3t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

19. Find a fundamental matrix for the system $x'(t) = Ax(t)$ for the given matrix A

$$A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} -1-r & 1 \\ 8 & 1-r \end{vmatrix} = (-1-r)(1-r) - 8 = (r^2 - 9) = (r-3)(r+3)$$

The eigenvalues are : $r = 3, -3$

For $r = 3$

$$(A - rI)u = \begin{bmatrix} -4 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4u_1 + u_2 = 0$
$8u_1 - 2u_2 = 0$
Solving the system$\Rightarrow$ $u_2 = 4u_1$
Letting $u_1 = s \Rightarrow u_2 = 4s$

$$\Rightarrow u_1 = \begin{bmatrix} s \\ 4s \end{bmatrix} = s \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

For $r = -3$

$$(A - rI)u = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2u_1 + u_2 = 0$
$8u_1 + 4u_2 = 0$
Solving the system$\Rightarrow$ $u_2 = -2u_1$
Letting $u_1 = s \Rightarrow u_2 = -2s$

$$\Rightarrow u_2 = \begin{bmatrix} s \\ -2s \end{bmatrix} = s\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus, the general solution for the system is:

$$x(t) = c_1e^{3t}u_1 + c_2e^{-3t}u_2 = c_1e^{3t}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus, the fundamental matrix for the system is:

$$\begin{bmatrix} e^{3t} & e^{-3t} \\ 4e^{3t} & -2e^{-3t} \end{bmatrix}$$

21. Find a fundamental matrix for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{bmatrix}$$

We first calculate the eigenvalues and eigenvectors.

$$\begin{vmatrix} -r & 1 & 0 \\ 0 & -r & 1 \\ 8 & -14 & 7-r \end{vmatrix} = -r^3 + 7r^2 - 14r + 8$$

Thus $p(r) = r^3 - 7r^2 + 14r - 8$. Note that $p(1) = p(2) = p(4) = 0$. Hence

$$p(r) = r^3 - 7r^2 + 14r - 8 = (r-1)(r-2)(r-4)$$

and the eigenvalues are $r = 1, 2, 4$.

The system $(A - rI)X = 0$ is

$$\begin{aligned} -rx_1 + x_2 + 0x_3 &= 0 \\ 0x_1 - rx_2 + x_3 &= 0 \\ 8x_1 - 14x_2 + (7-r)x_3 &= 0 \end{aligned}$$

$r = 1$ yields

$$\begin{aligned} -x_1 + x_2 &= 0 \\ -x_2 + x_3 &= 0 \\ 8x_1 - 14x_2 - 6x_3 &= 0 \end{aligned}$$

Thus $x_1 = x_2 = x_3$ and the corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $r = 2$ we get

$$-2x_1 + x_2 = 0$$
$$-2x_2 + x_3 = 0$$
$$8x_1 - 14x_2 + 5x_3 = 0$$

Thus $x_2 = 2x_1$ and $x_3 = 2x_2 = 4x_1$ and the corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

For $r = 4$ the system becomes

$$-4x_1 + x_2 + 0x_3 = 0$$
$$0x_1 - 4x_2 + x_3 = 0$$
$$8x_1 - 14x_2 + 3x_3 = 0$$

Thus $4x_1 = x_2$ and $x_3 = 4x_2$. Letting $x_1 = 1$ we have the eigenvector

$$\begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix}$$

Thus a fundamental matrix is

$$\begin{matrix} e^t & e^{2t} & e^{4t} \\ e^t & 2e^{2t} & 4e^{4t} \\ e^t & 4e^{2t} & 16e^{4t} \end{matrix}$$

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32. Solve the given IVP

$$\vec{x}'(t) = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}\vec{x}(t), \qquad \vec{x}(0) = \begin{bmatrix} -10 \\ 6 \end{bmatrix}.$$

$$\det(A - rI) = \begin{vmatrix} 6 - r & -3 \\ 2 & 1 - r \end{vmatrix} = (6 - r)(1 - r) + 6 = r^2 - 7r + 12 = 0$$

$$(r - 3)(r - 4)$$

$$\Rightarrow r = 3, 4$$

$$r = 3: \qquad u_1 = (A - 3I)u = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3u_1 - 3u_2 = 0$
$2u_1 - 2u_2 = 0$
Solving the system $\Rightarrow u_1 = u_2$
Setting $u_1 = s \Rightarrow u_2 = s$

$$\Rightarrow u_1 = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow 3u_1 = 3u_2 \Rightarrow u_1 = u_2$, so eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$r = 4: \qquad u_2 = (A - 4I)u = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2u_1 - 3u_2 = 0$
$2u_1 - 3u_2 = 0$
Solving the system $\Rightarrow u_1 = \frac{3}{2}u_2$
Setting $u_2 = s \Rightarrow u_1 = \frac{3}{2}s$

$$\Rightarrow u_2 = \begin{bmatrix} \frac{3}{2}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

A fundamental matrix is $\begin{bmatrix} e^{3t} & \frac{3}{2}e^{4t} \\ e^{3t} & e^{4t} \end{bmatrix}$.

Then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{3t} & \frac{3}{2}e^{4t} \\ e^{3t} & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\vec{x}(0) = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -10 \\ -6 \end{bmatrix}$$

$c_1 + \frac{3}{2}c_2 = -10$
$c_1 + c_2 = -6 \Rightarrow c_1 = -6 - c_2$

$\Rightarrow -6 - c_2 + \frac{3}{2}c_2 = -10 \Rightarrow c_2 = -8; c_1 = 2$

$$\text{So } \vec{x}(t) = \begin{bmatrix} e^{3t} & \frac{3}{2}e^{4t} \\ e^{3t} & e^{4t} \end{bmatrix} \begin{bmatrix} 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 2e^{3t} - 12e^{4t} \\ 2e^{3t} - 8e^{4t} \end{bmatrix}.$$

33. Solve the given IVP

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\vec{x}(t), \qquad \vec{x}(0) = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}.$$

$$\det(A - rI) = \begin{vmatrix} 1-r & -2 & 2 \\ -2 & 1-r & -2 \\ 2 & -2 & 1-r \end{vmatrix} = (1-r)\begin{vmatrix} 1-r & -2 \\ -2 & 1-r \end{vmatrix} + 2\begin{vmatrix} -2 & -2 \\ 2 & 1-r \end{vmatrix} + 2\begin{vmatrix} -2 & 1-r \\ 2 & -2 \end{vmatrix}$$

$$(1-r)\left[(1-r)^2 - 4\right] + 2[-2(1-r) + 4] + 2[4 - 2(1-r)] = 0$$
$$(1-r)(r-3)(r+1) + 8(r+1) = -(r+1)(r-5)(r+1)(r+1) = 0$$

$$\Rightarrow r = 5, -1(multiplicity 2)$$

$$r = 5 : \qquad u_1 = (A - 5I)u = \begin{bmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4u_1 - 2u_2 + 2u_3 = 0$
$-2u_1 - 4u_2 - 2u_3 = 0$
$2u_1 - 2u_2 - 4u_3 = 0$
Solving the system
$\Rightarrow 2u_1 = 2u_2 + 4u_3$
$-2u_2 - 4u_3 - 4u_2 - 2u_3 = -6u_2 - 6u_3 = 0$
$u_2 = -u_3$
$2u_1 = -2u_3 + 4u_3 \Rightarrow u_1 = u_3$
Setting $u_3 = s \Rightarrow u_2 = -s \Rightarrow u_1 = s$

$$\Rightarrow u_1 = \begin{bmatrix} s \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$r = -1 : \qquad u_1 = (A + 1I)u = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2u_1 - 2u_2 + 2u_3 = 0$
$-2u_1 + 2u_2 - 2u_3 = 0$
$2u_1 - 2u_2 + 2u_3 = 0$
Simplifying the system

$u_1 - u_2 + u_3 = 0$
$u_1 + u_2 - u_3 = 0$ (last 2 eqns are the same)

Setting $u_2 = s$ and $u_3 = v$ then $u_1 = s - v$

$$\Rightarrow u_2 = \begin{bmatrix} s - v \\ s \\ v \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The General Solutions is as follows:

$$x(t) = c_1 e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Plugging in IV

$$x(0) = c_1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}.$$

Solving the system:

$c_1 + c_2 - c_3 = -2$
$-c_1 + c_2 = -2$
$c_1 + c_3 = 2$
$\Rightarrow c_1 = 1, c_2 = -2, c_3 = 1$

$$x(t) = e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - 2e^{-t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$x(t) = \begin{bmatrix} e^{5t} - 2e^{-t} - e^{-t} \\ -e^{5t} - 2e^{-t} + 0 \\ e^{5t} + 0 + e^{-t} \end{bmatrix} = \begin{bmatrix} e^{5t} - 3e^{-t} \\ -e^{5t} - 2e^{-t} \\ e^{5t} + e^{-t} \end{bmatrix}$$