Ma 227 Homework due 11/1/2013

Page 971 #3, 5, 7, 9, 13
Section 13.8
3.Verify the divergence theorem is true for the vector field F on the region E.
$\vec{F} = \langle z, y, x \rangle$ E is the ball $x^2 + y^2 + z^2 \leq 161$.
$div\vec{F} = 0 + 1 + 0 = 1$ so

$$\int\int\int_E \operatorname{div} F dV = \iiint_E 1 dV = V(E) = \frac{4}{3}\pi 4^3$$

Let S is a sphere of radius 4 centered at the origin which can be parametrized by

$$\vec{r}(\phi, \theta) = \langle 4\sin\varphi\cos\theta. 4\cos\varphi\sin\theta, 4\cos\varphi \rangle$$

(similar to Example 12.6.1)

$$\begin{aligned}\vec{r}_\varphi \times \vec{r}_\theta &= \langle 4\cos\varphi\cos\theta, 4\cos\varphi\sin\theta, -4\sin\varphi \rangle \times \langle -4\sin\varphi\sin\theta, 4\sin\varphi\cos\theta, 0 \rangle \\ &= \left\langle 16\sin^2\varphi\cos\theta, 16\sin^2\varphi\sin\theta, 16\cos\varphi\sin\varphi \right\rangle\end{aligned}$$

and

$$\vec{F}(\vec{r}(\varphi, \theta)) = \langle 4\cos\varphi, 4\sin\varphi\sin\theta, 4\sin\varphi\cos\theta \rangle$$

$$\begin{aligned}\vec{F} \cdot (\vec{r}_\varphi \times \vec{r}_\theta) &= 64\cos\varphi\sin^2\varphi\cos\theta + 64\sin^3\varphi\sin^2\theta + 64\cos\varphi\sin^2\varphi\cos\theta \\ &= 128\cos\varphi\sin^2\varphi\cos\theta + 64\sin^3\varphi\sin^2\theta\end{aligned}$$

Then

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_D \vec{F} \cdot (\vec{r}_\varphi \times \vec{r}_\theta) dA = \int_0^{2\pi}\int_0^{\pi} \left(128\cos\varphi\sin^2\varphi\cos\theta + 64\sin^3\varphi\sin^2\theta\right) d\varphi d\theta \\ &= \int_0^{2\pi} \left[\frac{128}{3}\sin^3\varphi\cos\theta + 64\left(-\frac{1}{3}\left(2 + \sin^2\varphi\right)\cos\varphi\right)\sin^2\theta \right]_{\varphi=0}^{\varphi=\pi} d\theta \\ &= \int_0^{2\pi} \frac{256}{3}\sin^2\theta d\theta = \frac{256}{3}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \frac{256}{3}\pi\end{aligned}$$

5. $\operatorname{div}\vec{F} = \partial/\partial x(xye^z) + \partial/\partial y(xy^2z^3) + \partial/\partial z(-ye^z) = ye^z + 2xyz^3 - ye^z = 2xyz^3$
by the Divergence theorem,

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int\int_E \operatorname{div} F dV = \int_0^3\int_0^2\int_0^1 2xyz^3 dz dy dx = 2\left[\frac{1}{2}x^2 \right]_0^3 \left[\frac{1}{2}y^2 \right]_0^2 \left[\frac{1}{4}z^4 \right]_0^1 = \frac{9}{2}$$

7. $\vec{F}(x, y, z) = 3xy^2\vec{i} + xe^z\vec{j} + z^3\vec{k}$. S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$
$\operatorname{div} F = 3y^2 + 0 + 3z^2 = 3y^2 + 3z^2$
so using cylindrical coordinates where $y = r\cos\theta, z = r\sin\theta$ and $x = x$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (3y^2+3z^2)dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta) r\,dx\,dr\,d\theta = \frac{9}{2}\pi$$

9. $\vec{F}(x,y,z) = x^2\sin y\vec{i} + x\cos y\vec{j} - xz\sin y\vec{k}$. S is the "fat sphere" $x^8+y^8+z^8 = 8$
div $F = 2x\sin y - x\sin y - x\sin y = 0$ so by Divergence The0rem

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 0\,dV = 0$$

13. $\vec{F}(x,y,z) = 4x^3z\vec{i} + 4y^3ze^z\vec{j} + 3z^4\vec{k}$. S is the sphere with radius R and center the origin.
div $F = 12x^2z + 12y^2z + 12z^3$ so using spherical coordinates

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 12z(x^2+y^2+z^2)dV = \int_0^{2\pi}\int_0^{\pi}\int_{-1}^{R} 12(\rho\cos\phi)\left(\rho^2\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= 12\int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\cos\phi\,d\phi\int_0^R \rho^5 d\rho = 12(2\pi)\left[\frac{1}{2}\sin^2\phi\right]_0^{\pi}\left[\frac{1}{6}\rho^6\right]_0^R = 0$$

9.2 #1, 5, 7, 9 (not assigned) 12a, 13

1. $\begin{matrix} x_1+2x_2+2x_3 = 6 \\ 2x_1+x_2+x_3 = 6 \\ x_1+x_2+3x_3 = 6 \end{matrix} = \begin{bmatrix} 1 & 2 & 2 & 6 \\ 2 & 1 & 1 & 6 \\ 1 & 1 & 3 & 6 \end{bmatrix}$

$R_2 - 2R_1 \quad and \quad R_3 - R_1 \begin{bmatrix} 1 & 2 & 2 & 6 \\ 0 & -3 & -3 & -6 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

$\frac{-1}{3}R_2 \begin{bmatrix} 1 & 2 & 2 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

$R_1 - 2R_2 \;\; and \;\; R_3 + R_2 \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

$x_1 = 2 \quad x_2 + x_3 = 2 \quad 2x_3 = 2 \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

5. $\begin{matrix} -x_1+2x_2 = 0 \\ 2x_1+3x_2 = 0 \end{matrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 0 \end{bmatrix}$

$R_2 + 2R_1 : \begin{bmatrix} -1 & 2 & 0 \\ 0 & 7 & 0 \end{bmatrix} = \begin{matrix} -x_1+2x_2 = 0 \\ 7x_2 = 0 \end{matrix}$

So x_2=0. Substituting 0 in for x_2 for the first equation gives x_1=0.

9.

We eliminate x_1 from the first equation by adding (1-i) times the second equation to it:

$$[2-(1+i)(1-i)]x_2 = 0$$
$$-x_1-(1+i)x_2 = 0$$

Since (1-i)(1+i)=$1^2 - i^2 = 1-(-1) = 2,$ we obtain

$$\begin{array}{r} 0 = 0 \\ -x_1-(1+i)x_2 = 0 \end{array} \quad \Rightarrow x_2 = \frac{-1}{1+i}x_1 = \frac{-1+i}{2}x_1$$

Assigning an arbitrary complex value to x_1, say 2s, we see that the system has infinitely many solutions given by $x_1 = 2s$ $x_2 = (-1+i)s$ where s is any complex number.

12a.

$$2x_1 - x_2 = 2$$
$$-6x_1 + 3x_2 = 4$$

Multiplying the first equation by –3 yields $-6x_1 + 3x_2 = -6.$ Both this equation and the second equation above cannot hold at the same time.

13.

$$2x_1 - 3x_2 = rx_1$$
$$x_1 - 2x_2 = rx_2$$

Or

$$(2-r)x_1 - 3x_2 = 0$$
$$x_1 + (-r-2)x_2 = 0$$

For $r = 2$ we have

$$-3x_1 = 0$$
$$x_1 - 4x_2 = 0$$

so the only solution is $x_1 = x_2 = 0.$

For $r = 1$ we have

$$x_1 - 3x_2 = 0$$
$$x_1 - 3x_2 = 0$$

Thus $x_1 = 3x_2$ So letting $x_2 = t,$ we have $x = 3t,$ and $x_2 = t,$ where t can have any value and thus we have an infinite number of solutions.