Ma 227 Homework due 11/8/2013

9.3 p. 515 #1, 3, 6, 7, 9, 10, 16, 19, 23, 28

1.

a) $A + B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix}$

b) $3A - B = 3\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 7 & 18 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$.

a) $AB = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -2+20 & 6+8 \\ -1+5 & 3+2 \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 4 & 5 \end{bmatrix}$

b) $A^2 = AA = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4+4 & 8+4 \\ 2+1 & 4+1 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix}$

c) $B^2 = BB = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1+15 & -3+6 \\ -5+10 & 15+4 \end{bmatrix} = \begin{bmatrix} 16 & 3 \\ 5 & 19 \end{bmatrix}$

6.

a.) $AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0+2 & 3+4 \\ 0+1 & 3+2 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$

b.) $(AB)C = \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+7 & -8+7 \\ 1+5 & -4+5 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ 6 & 1 \end{bmatrix}$

c.) $(A+B)C = \left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}\right)\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+5 & -4+5 \\ 2+3 & -8+3 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 5 & -5 \end{bmatrix}$

7.

a.) $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots\ldots\ldots u_n v_n$

$\mathbf{u}^T = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \ldots\ldots\ldots u_nv_n$$

b.)

$$\mathbf{v}^T = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad A\mathbf{v} = \begin{bmatrix} 1 & 2 & 6 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2+6+30 \\ -2+6-5 \end{bmatrix} = \begin{bmatrix} 38 \\ -1 \end{bmatrix}$$

$$(A\mathbf{v})^T = \begin{bmatrix} 38 & -1 \end{bmatrix} \quad \mathbf{v}^TA^T = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 2+6+30 & -2+6-5 \end{bmatrix} = \begin{bmatrix} 38 & -1 \end{bmatrix}$$

c.) $(A\mathbf{v})^T = \mathbf{v}^TA^T$ $\quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ A, transpose:

$$\begin{bmatrix} a_{11} & a_{21} & \vdots & a_{m1} \\ a_{12} & a_{22} & \vdots & a_{m2} \\ \cdots & \ldots & \cdots & \cdots \\ a_{1n} & a_{2n} & \vdots & a_{mn} \end{bmatrix}$$

$A\mathbf{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots a_{mn}v_n \end{bmatrix}$ $A\mathbf{v}$, transpose:

$$\begin{bmatrix} v_1a_{11} + v_2a_{12} + \cdots a_{1n}v_n & v_1a_{21} + v_2a_{22} + \cdots v_na_{2n} & \vdots & v_1a_{m1} + \cdots v_na_{mn} + v_2a_{2m} \end{bmatrix}$$

$$\mathbf{v}^TA^T =: \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & \vdots & a_m \\ a_{12} & a_{22} & \vdots & a_{2m} \\ \cdots & \ldots & \cdots & \cdots \\ a_{1n} & a_{2n} & \vdots & a_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} v_1a_{11} + v_2a_{12} + \cdots a1_nv_n & v_1a_{21} + v_2a_{22} + \cdots v_na_{2n} & \vdots & v_1a_{m1} + \cdots v_na_{mn} + v_2a_{m2} \end{bmatrix}$$

d.) $A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$, transpose: $\begin{bmatrix} a_{11} & a_{21} & a_{m1} \\ a_{12} & a_{22} & a_{m2} \\ a_{1n} & a_{2n} & a_{mn} \end{bmatrix}$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{1n} \\ b_{21} & b_{22} & b_{2n} \\ b_{m1} & b_{m2} & b_{mn} \end{bmatrix}, \text{ transpose: } \begin{bmatrix} b_{11} & b_{21} & b_{m1} \\ b_{12} & b_{22} & b_{m2} \\ b_{1n} & b_{2n} & b_{mn} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{1n} \\ b_{21} & b_{22} & b_{2n} \\ b_{m1} & b_{m2} & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots a_{1n}b_{m1} & a_{11}b_{12} + a_{12}b_{22} + \cdots a_{1n}b_{m2} & a_{11}b_{1n} + a_{12}b_{2n} + \cdots a_{1n}b_{mn} \\ a_{21}b_{11} + a_{22}b_{21} + \ldots b_{m1}a_{2n} & a_{21}b_{12} + a_{22}b_{22} + \cdots a_{2n}b_{m2} & a_{21}b_{1n} + a_{22}b_{2n} + \cdots b_{mn}a_{2n} \\ b_{11}a_{m1} + b_{21}a_{m2} + \cdots b_{m1}a_{mn} & b_{12}a_{m1} + b_{22}a_{m2} + \cdots a_{mn}b_{m2} & a_{1m}b_{1n} + a_{m2}b_{2n} + \cdots a_{mn}b_{mn} \end{bmatrix}$$

,

$(AB)^T$:

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots a_{1n}b_{m1} & a_{21}b_{11} + a_{22}b_{21} + \ldots b_{m1}a_{2n} & b_{11}a_{m1} + b_{21}a_{m2} + \cdots b_{m1}a_{mn} \\ a_{11}b_{12} + a_{12}b_{22} + \cdots a_{1n}b_{m2} & a_{21}b_{12} + a_{22}b_{22} + \cdots a_{2n}b_{m2} & b_{12}a_{m1} + b_{22}a_{m2} + \cdots a_{mn}b_{m2} \\ a_{11}b_{1n} + a_{12}b_{2n} + \cdots a_{1n}b_{mn} & a_{21}b_{1n} + a_{22}b_{2n} + \cdots b_{mn}a_{2n} & a_{1m}b_{1n} + a_{m2}b_{2n} + \cdots a_{mn}b_{mn} \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} b_{11} & b_{21} & b_m \\ b_{12} & b_{22} & b_{2m} \\ b_n & b_{2n} & b_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_m \\ a_{12} & a_{22} & a_{2m} \\ a_n & a_{2n} & a_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots a_{1n}b_{m1} & a_{21}b_{11} + a_{22}b_{21} + \ldots b_{m1}a_{2n} & b_{11}a_{m1} + b_{21}a_{m2} + \cdots b_{m1}a_{mn} \\ a_{11}b_{12} + a_{12}b_{22} + \cdots a_{1n}b_{m2} & a_{21}b_{12} + a_{22}b_{22} + \cdots a_{2n}b_{m2} & b_{12}a_{m1} + b_{22}a_{m2} + \cdots a_{mn}b_{m2} \\ a_{11}b_{1n} + a_{12}b_{2n} + \cdots a_{1n}b_{mn} & a_{21}b_{1n} + a_{22}b_{2n} + \cdots b_{mn}a_{2n} & a_{1m}b_{1n} + a_{m2}b_{2n} + \cdots a_{mn}b_{mn} \end{bmatrix}$$

9. $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \quad |A| = (2)(4) - (1)(-1) = 8 + 1 = 9$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{-1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}$$

10. Let $A = \begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix}$, Find A^{-1}

$$\begin{bmatrix} 4 & 1 & 1 & 0 \\ 5 & 9 & 0 & 1 \end{bmatrix}$$

$$R_2 - \frac{5}{4}R_1 : \begin{bmatrix} 4 & 1 & 1 & 0 \\ 0 & \frac{31}{4} & \frac{-5}{4} & 1 \end{bmatrix}$$

$$R_1 - \frac{4}{31}R_2 : \begin{bmatrix} 4 & 0 & \frac{36}{31} & \frac{-4}{31} \\ 0 & \frac{31}{4} & \frac{-5}{4} & 1 \end{bmatrix}$$

$\frac{1}{4}R_1$: $\begin{bmatrix} 1 & 0 & \frac{9}{31} & \frac{-1}{31} \\ 0 & \frac{31}{4} & \frac{-5}{4} & 1 \end{bmatrix}$

$\frac{4}{31}R_2$: $\begin{bmatrix} 1 & 0 & \frac{9}{31} & \frac{-1}{31} \\ 0 & 1 & \frac{-5}{31} & \frac{4}{31} \end{bmatrix}$

16. a) Show that $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is singular; ie, show that A has no inverse.

Augmented matrix: $\begin{bmatrix} 2 & -1 & 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$.

$\frac{1}{2}R_1$: $\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$

R_2 by $R_2 + R_1$; R_3 by $R_3 - R_1$:

$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$

$\frac{2}{3}R_2$: $\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$

R_1 by $R_1 + \frac{1}{2}R_2$; R_3 by $R_3 - \frac{3}{2}R_2$:

$\begin{bmatrix} 1 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$

But $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \neq I$

OR: $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, determinant: 0 $\Rightarrow$ matrix is singular.

b) Show that $Ax = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ has no solutions.

Set up the augmented matrix: $\begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}$. Now use row operations:

$$\frac{1}{2}R_1 : \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ -1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$R_2 \text{ by } R_2 + R_1; R_3 \text{ by } R_3 - R_1 : \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{5}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

$$\frac{2}{3}R_2 : \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

$$R_1 \text{ by } R_1 + \frac{1}{2}R_2; R_3 \text{ by } R_3 - \frac{3}{2}R_2 : \begin{bmatrix} 1 & 0 & 1 & \frac{7}{3} \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Last row is saying $0x_1 + 0x_2 + 0x_3 = -1$, which is of course impossible, so there are no solutions.

c) Show that $Ax = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$ has infinitely many solutions.

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 3 \end{bmatrix}. \qquad \frac{1}{2}R_1 : \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$R_2 \text{ by } R_2 + R_1; R_3 \text{ by } R_3 - R_1 : \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

$$\frac{2}{3}R_2 : \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

R_1 by $R_1 + \frac{1}{2}R_2; R_3$ by $R_3 - \frac{3}{2}R_2$: $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now if we let $x_3 = c, c$ arbitrary, then we have $x_2 = 1 - c, x_1 = 2 - c$.

Can also express solution as $x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

19. $X(t) = \begin{bmatrix} e^t & e^{-t} & e^{2t} \\ e^t & -e^{-t} & 2e^{2t} \\ e^t & e^{-t} & 4e^{2t} \end{bmatrix}$. $X^{-1}(t) = ?$

$\begin{bmatrix} e^t & e^{-t} & e^{2t} & 1 & 0 & 0 \\ e^t & -e^{-t} & 2e^{2t} & 0 & 1 & 0 \\ e^t & e^{-t} & 4e^{2t} & 0 & 0 & 1 \end{bmatrix}$. R_2 by $R_2 - R_1; R_3$ by $R_3 - R_1$:

$\begin{bmatrix} e^t & e^{-t} & e^{2t} & 1 & 0 & 0 \\ 0 & -2e^{-t} & e^{2t} & -1 & 1 & 0 \\ 0 & 0 & 3e^{2t} & -1 & 0 & 1 \end{bmatrix}$

$e^{-t}R_1$: $\begin{bmatrix} 1 & e^{-2t} & e^t & e^{-t} & 0 & 0 \\ 0 & -2e^{-t} & e^{2t} & -1 & 1 & 0 \\ 0 & 0 & 3e^{2t} & -1 & 0 & 1 \end{bmatrix}$

$-\frac{1}{2}e^tR_2; \frac{1}{3}e^{-2t}R_3$: $\begin{bmatrix} 1 & e^{-2t} & e^t & e^{-t} & 0 & 0 \\ 0 & 1 & -\frac{1}{2}e^{3t} & \frac{1}{2}e^t & -\frac{1}{2}e^t & 0 \\ 0 & 0 & 1 & -\frac{1}{3}e^{-2t} & 0 & \frac{1}{3}e^{-2t} \end{bmatrix}$

R_1 by $R_1 - e^{-2t}R_2$: $\begin{bmatrix} 1 & 0 & \frac{3}{2}e^t & \frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} & 0 \\ 0 & 1 & -\frac{1}{2}e^{3t} & \frac{1}{2}e^t & -\frac{1}{2}e^t & 0 \\ 0 & 0 & 1 & -\frac{1}{3}e^{-2t} & 0 & \frac{1}{3}e^{-2t} \end{bmatrix}$

R_1 by $R_1 - \frac{3}{2}e^tR_3; R_2$ by $R_2 + \frac{1}{2}e^{3t}R_3$: $\begin{bmatrix} 1 & 0 & 0 & e^{-t} & \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ 0 & 1 & 0 & \frac{1}{3}e^t & -\frac{1}{2}e^t & \frac{1}{6}e^t \\ 0 & 0 & 1 & -\frac{1}{3}e^{-2t} & 0 & \frac{1}{3}e^{-2t} \end{bmatrix}$

$$\Rightarrow X^{-1}(t) = \begin{bmatrix} e^{-t} & \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ \frac{1}{3}e^{t} & -\frac{1}{2}e^{t} & \frac{1}{6}e^{t} \\ -\frac{1}{3}e^{-2t} & 0 & \frac{1}{3}e^{-2t} \end{bmatrix}.$$

23. Find determinant: $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & -2 \end{bmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & -2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} - 0\begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} + 0\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 1(-2-10) - 0 + 0 = -12.$$

Or $\begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & -2 \end{vmatrix}\begin{matrix} 1 & 0 \\ 3 & 1 \\ 1 & 5 \end{matrix} = (1)(1)(-2) + 0 + 0 - 0 - (5)(2)(1) - 0 = -12$

28. Determine the values of r for which $\det(A - rI) = 0$.

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix} \Rightarrow \det(A - rI) = \begin{vmatrix} 3-r & 3 \\ 2 & 4-r \end{vmatrix} = (3-r)(4-r) - 6 = 0$$

$$\Rightarrow 12 - 7r + r^2 - 6 = 0$$

$$r^2 - 7r + 6 = 0$$

$$r = 6; r = 1$$

9.5 - 3, ,5, 7, 11, 13, 19

3. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

$$\det(A - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & -1 \\ 2 & 4-r \end{vmatrix} = (1-r)(4-r) + 2 = 6 - 5r + r^2$$

$$= (r-3)(r-2) = 0 \Rightarrow r = 3, 2.$$

So the eigenvalues are 3 & 2.

Now, for each eigenvalue, solve $(A - rI)u = 0$:

$r = 3$: $\quad A - rI = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$;

$(A - rI)u = 0 \Rightarrow \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Both equations imply $u_2 = -2u_1$.

Let $u_1 = s$; then $u_2 = -2s$, and the associated eigenvector is $\begin{bmatrix} s \\ -2s \end{bmatrix}$ or $s\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$r = 2$: $\quad A - rI = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$;

$(A - rI)u = 0 \Rightarrow \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Both equations imply $u_2 = -u_1$.

Let $u_1 = s$; then $u_2 = -s$, and the associated eigenvector is $\begin{bmatrix} s \\ -s \end{bmatrix}$ or $s\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

5. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$.

$$\det(A - rI) = \begin{vmatrix} 1-r & 0 & 0 \\ 0 & -r & 2 \\ 0 & 2 & -r \end{vmatrix} = (1-r)\begin{vmatrix} -r & 2 \\ 2 & -r \end{vmatrix} - 0 + 0 = (1-r)(r^2 - 4) = (1-r)(r-2)(r+2)$$

Eigenvalues:$\Rightarrow$ $r = 1, 2, -2$

Now, for each eigenvalue, solve $(A - rI)u = 0$:

$r = 1$: $\quad A - rI = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

$(A - rI)u = 0 \Rightarrow$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-u_2 + 2u_3 = 0$
$2u_2 - u_3 = 0$
Solving the system of equations$\Rightarrow$ $u_2 = u_3 = 0$
We assign any value to u_1 say $u_1 = s$. Therefore:

$\Rightarrow$The associated eigenvector for $r = 1$ is $\begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$ or $s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$r = 2$: $\quad A - rI = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix}$

$(A - rI)u = 0 \Rightarrow$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-u_1 = 0$
$-2u_2 + 2u_3 = 0$
$2u_2 - 2u_3 = 0$
Solving the system of equations$\Rightarrow$ $u_1 = 0; u_2 = u_3$
We assign any value to u_2 say $u_2 = u_3 = s$. Therefore:

$\Rightarrow$The associated eigenvector for $r = 2$ is $\begin{bmatrix} 0 \\ s \\ s \end{bmatrix}$ or $s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$r = -2$: $\quad A - rI = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

$(A - rI)u = 0 \Rightarrow$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3u_1 = 0$
$2u_2 + 2u_3 = 0$
$2u_2 + 2u_3 = 0$

Solving the system of equations$\Rightarrow$ $u_1 = 0; u_2 = -u_3$
We assign any value to u_3,say $u_3 = s$. Then $u_2 = -s$

$\Rightarrow$The associated eigenvector for $r = -2$ is $\begin{bmatrix} 0 \\ -s \\ s \end{bmatrix}$ or $s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

7. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix}$

$$\det(A - rI) = \begin{vmatrix} 1-r & 0 & 0 \\ 2 & 3-r & 1 \\ 0 & 2 & 4-r \end{vmatrix} = 8r^2 - 17r - r^3 + 10 = -(r-5)(r-1)(r-2)$$

$-(r-5)(r-1)(r-2) = 0$
Eigenvalues: $r = 1, 2, 5$
Eigenvectors:
$r = 1$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \end{bmatrix} R_1 \to R_2; R_2 \to R_3 : \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R_1 by $R_1 - R_2$: $\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow u_1 = u_3; 2u_2 = -3u_3$

Let $u_1 = s$. Then $u_2 = -\frac{3}{2}s = s \Rightarrow \vec{u} = s\begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$

$r = 2$:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} R_1 \to -R_1; R_2 \text{ by } R_2 - 2R_1; \tfrac{1}{2}R_3 : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

R_3 by $R_3 - R_2$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow u_1 = 0; u_2 = -u_3$

Let $u_3 = s$. Then $u_2 = -s \Rightarrow \vec{u} = s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$r = 5$:

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} R_1 \to -\tfrac{1}{4}R_1; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

R_2 by $R_2 + R_3$; $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$; R_2 by $R_2 - 2R_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \Rightarrow u_1 = 0; u_3 = 2u_2$

Let $u_2 = s$. Then $u_3 = 2s \Rightarrow \vec{u} = s\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

11. Find a general solution to the system

$$A = \begin{bmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{bmatrix}$$

$$|A - rI| = \begin{bmatrix} -1-r & \frac{3}{4} \\ -5 & 3-r \end{bmatrix} = 0$$

$$r^2 - 2r + \tfrac{3}{4} = 0 \Rightarrow r = \tfrac{1}{2}, \tfrac{3}{2} \Rightarrow \text{eigenvalues}$$

$$\left(A - \tfrac{1}{2}r\right)u = \begin{bmatrix} -\frac{3}{2} & \frac{3}{4} \\ -5 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -\tfrac{3}{2}u_1 + \tfrac{3}{4}u_2 &= 0 \\ -5u_1 + \tfrac{5}{2}u_2 &= 0 \end{aligned} \Rightarrow u_1 = \tfrac{1}{2}u_2 \Rightarrow u_2 = s; u_1 = \tfrac{1}{2}s$$

$$u = s\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow eigenvector$$

$$\left(A - \tfrac{3}{2}r\right)u = \begin{bmatrix} -\frac{5}{2} & \frac{3}{4} \\ -5 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} -\frac{5}{2}u_1 + \frac{3}{4}u_2 = 0 \\ -5u_1 + \frac{3}{2}u_2 = 0 \end{array} \Rightarrow u_1 = \frac{6}{20}u_2 \Rightarrow u_2 = s; u_1 = \frac{3}{10}s$$

$$u = s\begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix} \Rightarrow \text{eigenvector}$$

General Solution: $X(t)c = c_1 e^{\frac{1}{2}t}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{\frac{3}{2}t}\begin{bmatrix} \frac{3}{10} \\ 1 \end{bmatrix}$

13. Find a general solution to the system

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} 1-r & 2 & 2 \\ 2 & -r & 3 \\ 2 & 3 & -r \end{vmatrix} = 0$$

$$\Rightarrow (1-r)\begin{vmatrix} -r & 3 \\ 3 & -r \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 2 & -r \end{vmatrix} + 2\begin{vmatrix} 2 & -r \\ 2 & 3 \end{vmatrix} = 0$$

$\Rightarrow (1-r)(r^2 - 9) - 2(-2r - 6) + 2(6 + 2r) = (1-r)(r-2)(r+2) = 0$
$\Rightarrow (r+3)[(1-r)(r-3) + 8] = 0 \Rightarrow (r+3)(r-5)(r+1) = 0$
The eigenvalues are:
$r = -3, 5, -1$

Eigenvector for:

$r = -3$

$$(A + 3I)u = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4u_1 + 2u_2 + 2u_3 = 0$
$2u_1 + 3u_2 + 3u_3 = 0$
$2u_1 + 3u_2 + 3u_3 = 0$
Solving the system $\Rightarrow u_1 = 0; u_2 = -u_3;$
Setting $u_3 = s \Rightarrow u_2 = -s$

$$\Rightarrow u_1 = \begin{bmatrix} 0 \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$r = 5$

$$(A + 3I)u = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -5 & 3 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4u_1 + 2u_2 + 2u_3 = 0$
$2u_1 - 5u_2 + 3u_3 = 0$
$2u_1 + 3u_2 - 5u_3 = 0$
Solving the system$\Rightarrow u_1 = u_3, u_2 = u_3$
Letting $u_3 = s \Rightarrow u_1 = u_2 = u_3 = s$

$$\Rightarrow u_2 = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$r = -1$

$$(A + 3I)u = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2u_1 + 2u_2 + 2u_3 = 0$

$2u_1 + u_2 + 3u_3 = 0$
$2u_1 + 3u_2 + 1u_3 = 0$
Solving the system$\Rightarrow u_1 = -2u_2, u_3 = u_2$
Letting $u_2 = s \Rightarrow u_1 = -2s, u_3 = s$

$$\Rightarrow u_3 = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the general solution for the system is:

$$x(t) = c_1 e^{-3t}u_1 + c_2 e^{-t}u_2 + c_3 e^{5t}u_3 = c_1 e^{-3t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

19. Find a fundamental matrix for the system $x'(t) = Ax(t)$ for the given matrix A

$$A = \begin{bmatrix} -1 & 1 \\ 8 & 1 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} -1-r & 1 \\ 8 & 1-r \end{vmatrix} = (-1-r)(1-r) - 8 = (r^2 - 9) = (r-3)(r+3)$$

The eigenvalues are : $r = 3, -3$

For $r = 3$

$$(A - rI)u = \begin{bmatrix} -4 & 1 \\ 8 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4u_1 + u_2 = 0$
$8u_1 - 2u_2 = 0$
Solving the system$\Rightarrow u_2 = 4u_1$
Letting $u_1 = s \Rightarrow u_2 = 4s$

$$\Rightarrow u_1 = \begin{bmatrix} s \\ 4s \end{bmatrix} = s\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

For $r = -3$

$$(A - rI)u = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2u_1 + u_2 = 0$
$8u_1 + 4u_2 = 0$
Solving the system$\Rightarrow u_2 = -2u_1$
Letting $u_1 = s \Rightarrow u_2 = -2s$

$$\Rightarrow u_2 = \begin{bmatrix} s \\ -2s \end{bmatrix} = s\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus, the general solution for the system is:

$$x(t) = c_1e^{3t}u_1 + c_2e^{-3t}u_2 = c_1e^{3t}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus, the fundamental matrix for the system is:

$$\begin{bmatrix} e^{3t} & e^{-3t} \\ 4e^{3t} & -2e^{-3t} \end{bmatrix}$$