Ma 227 Homework Solutions Fall 2013
Due 12/6/2013

9.7 - 1, 3,7,9

1. Use undetermined coefficients to find a general solution to the system $\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t)$.

$A = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}, \vec{f}(t) = \begin{bmatrix} -11 \\ -5 \end{bmatrix}$,

eigenvectors: $\left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 7$

If you solve the homogeneous case first, you'll find that

$$\vec{x}_h(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since the entries in $f(t)$ are just linear functions of t, we are inclined to seek a particular solution in the form:

$$\vec{x}_p(t) = \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Then $\vec{x}_p'(t) = A\vec{x}_p(t) + \vec{f}(t)$ gives

$$-\begin{bmatrix} -11 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} ta_1 \\ ta_2 \end{bmatrix}$$

$$-\begin{bmatrix} -11 \\ -5 \end{bmatrix} = \begin{bmatrix} 6a_1 & a_2 \\ 4a_1 & 3a_2 \end{bmatrix}$$

$6a_1 + a_2 = 11$
$4a_1 + 3a_2 = 5$

$4a_1 + 3(11 - 6a_1) = 5 \Rightarrow 4a_1 + 33 - 18a_1 = 5 \Rightarrow 28 = 14a_1$
$a_1 = 2; a_2 = -1$

Therefore:

$$\vec{x}_g(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \vec{f}(t) = \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}.$

If you solve the homogeneous case first, you'll find that

$$\vec{x}_h(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Since e^t does not appear in the homogeneous solution, our guess is simply

$$\vec{x}_p(t) = \vec{a}e^t = e^t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$ Then $\vec{x}_p'(t) = A\vec{x}_p(t) + \vec{f}(t)$ gives

Then $\vec{x}_p'(t) = A\vec{x}_p(t) + \vec{f}(t)$ gives

$$\begin{bmatrix} a_1e^t \\ a_2e^t \\ a_3e^t \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1e^t \\ a_2e^t \\ a_3e^t \end{bmatrix} + \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1e^t \\ a_2e^t \\ a_3e^t \end{bmatrix} - \begin{bmatrix} a_1e^t \\ a_2e^t \\ a_3e^t \end{bmatrix}$$

On the right, multiply out and combine to get

$$\begin{bmatrix} 2e^t \\ 4e^t \\ -2e^t \end{bmatrix} = \begin{bmatrix} a_1e^t - 2a_2e^t + 2a_3e^t - a_1e^t \\ -2a_1e^t + a_2e^t + 2a_3e^t - a_2e^t \\ 2a_1e^t + 2a_2e^t + a_3e^t - a_3e^t \end{bmatrix} = \begin{bmatrix} -2a_2e^t + 2a_3e^t \\ -2a_1e^t + 2a_3e^t \\ 2a_1e^t + 2a_2e^t \end{bmatrix}$$

$\Rightarrow e^t \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t$; this leads to the following system of 3 equations in 3 unknowns:

$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. Again, solve it any way you want; the augmented matrix

is $\begin{bmatrix} 0 & -2 & 2 & 2 \\ -2 & 0 & 2 & 4 \\ 2 & 2 & 0 & -2 \end{bmatrix}$. Row operations lead to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \vec{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \vec{x}_p(t) = e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\Rightarrow \vec{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Use undetermined coefficients to determine only the form of a particular solution for the system $\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t)$.

7. $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \vec{f}(t) = \begin{bmatrix} \sin 3t \\ t \end{bmatrix}.$

$\vec{f}(t)$ is a vector containing both a sine function and a polynomial of degree 1. Our guess must encompass both of these things. Guess: $\vec{x}_p(t) = \vec{a}t + \vec{b} + \sin 3t\vec{c} + \cos 3t\vec{d}.$

9.8 - 2,3,5

2. $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

$$\det(A - rI) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-2)^2 = 0$$

$r = 2$ is an eigenvalue of multiplicity 2 ($k = 2$). By Cayley-Hamilton, $(A - 2I)^2 = 0$, and

$$e^{At} = e^{2t}(I + (A - 2I)t)$$
$$= e^{2t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} t\right)$$
$$= e^{2t}\begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}$$

3. $A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}.$

$$\det(A - rI) = \begin{vmatrix} 2-r & 1 & -1 \\ -3 & -1-r & 1 \\ 9 & 3 & -4-r \end{vmatrix} = -(r+1)^3 = 0$$

$r = -1$ is an eigenvalue of multiplicity 3 ($k = 3$). By Cayley-Hamilton, $(A + I)^3 = 0$, and

$$e^{At} = e^{-t}\left(I + (A+I)t + (A+I)^2 \frac{t^2}{2}\right)$$

$$= e^{-t}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -1 \\ -3 & 0 & 1 \\ 9 & 1 & -3 \end{bmatrix} t + \begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & 0 & 3 \end{bmatrix} \frac{t^2}{2}\right)$$

$$= e^{-t}\begin{bmatrix} 1 + 3t - \frac{3}{2}t^2 & t & -t + \frac{t^2}{2} \\ -3t & 1 & t \\ 9t - \frac{9}{2}t^2 & 3t & 1 - 3t + \frac{3}{2}t^2 \end{bmatrix}.$$

5. $A = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

$$\det(A - rI) = \begin{vmatrix} -2 - r & 0 & 0 \\ 4 & -2 - r & 0 \\ 1 & 0 & -2 - r \end{vmatrix} = -r^3 - 6r^2 - 12r - 8 = -(r + 2)^3$$

$r = -2$ is an eigenvalue of multiplicity 3 ($k = 3$). By Cayley-Hamilton, $(A + 2I)^3 = 0$,

Check: $A + 2I = \begin{bmatrix} -2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ so $\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Thus $\quad e^{At} = e^{-2t}\left(I + (A + 2I)t + (A + 2I)^2 \frac{t^2}{2}\right)$

$$= e^{-2t}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{t^2}{2}\right)$$

$$= e^{-2t}\begin{bmatrix} 1 & 0 & 0 \\ 4t & 1 & 0 \\ t & 0 & 1 \end{bmatrix}.$$

9. Use the method of undetermined coefficients to determine only the form of the particular solution for the system

$$x'(t) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} e^{2t} \\ \sin t \\ t \end{bmatrix}$$

Since

$$\begin{bmatrix} e^{2t} \\ \sin t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin t \\ 0 \end{bmatrix}$$

then we need a particular solution for each piece of this. Hence

$$x_p(t) = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \\ a_3 t + b_3 \end{bmatrix} + \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t} \\ c_3 e^{2t} \end{bmatrix} + \begin{bmatrix} d_1 \sin t + e_1 \cos t \\ d_2 \sin t + e_2 \cos t \\ d_2 \sin t + e_2 \cos t \end{bmatrix}.$$