Ma 227 Homework 6 Solutions Fall 2010
Due 10/21/2010

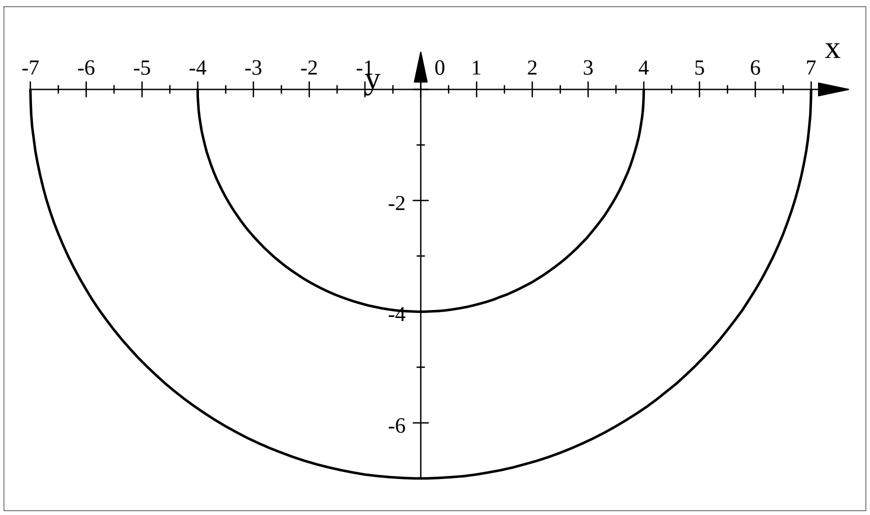
12.4-pg. 857 #3, 5, 7, 9, 11, 15, 21, 23, 25, 29

3. The region R is more easily described by the rectangular coordinates:
$R = \left\{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le \frac{1}{2}x + \frac{1}{2}\right\}$

Thus $\int\int_R f(x,y)dA = \int_{-1}^{1}\int_0^{\frac{1}{2}x+\frac{1}{2}} f(x,y)dydx$

5. $\int_\pi^{2\pi}\int_4^7 rdrd\theta$ represents the area of the region $R = \left\{(r,\theta) \mid 4 \le r \le 7,\ \pi \le \theta \le 2\pi\right\}$, the lower half of a ring as shown below.

$r = 4$



$$\int_\pi^{2\pi}\int_4^7 rdrd\theta = \int_\pi^{2\pi}\frac{r^2}{2}\Big]_4^7 d\theta = \pi\left(\frac{1}{2}\right)(49-16) = \frac{33\pi}{2}$$

7. $\int\int_D xydA,$ where D is the disk with center at the origin and radius 3.
D can be described as $D = \left\{(r,\theta) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi\right\}$

$\Rightarrow$

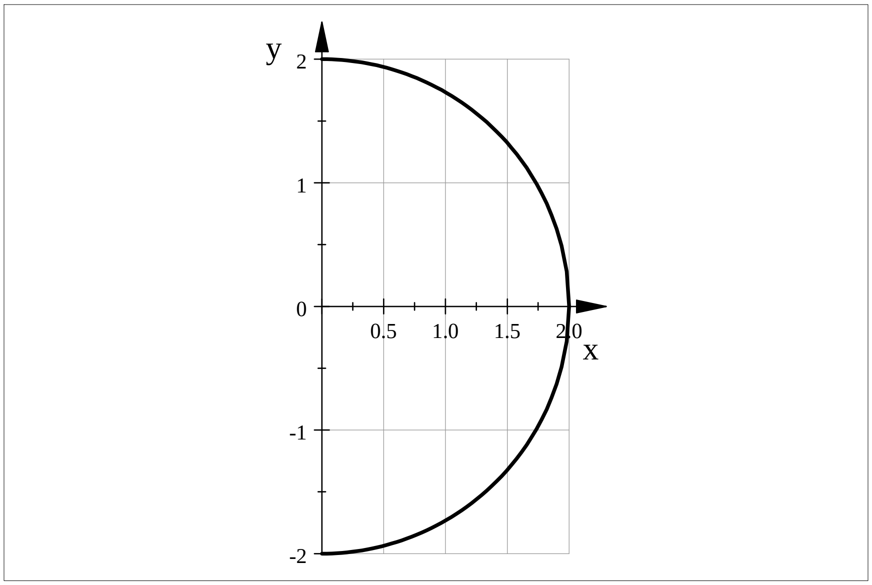
$$\int\int_D xydA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)rdrd\theta = \int_0^{2\pi}\int_0^3 r^3\cos\theta\sin\theta drd\theta$$
$$= \int_0^{2\pi}\frac{r^4}{4}\Big]_0^3\cos\theta\sin\theta drd\theta = \frac{81}{4}\left(\frac{1}{2}\sin^2\theta\right)_0^{2\pi} = 0$$

9. $\int\int_R \cos(x^2+y^2)dA$ where R is the region that lies above the x-axis within the circle $x^2+y^2 = 9$

D can be described as $D = \{(r, \theta) | \ \le r \le, \ 0 \le \theta \le \}$

11. $\iint\limits_D e^{-x^2-y^2} dA.$ The region of integration is shown below.

$\sqrt{4-y^2}$



Switching to polar coordinates we have

$$\iint\limits_D e^{-x^2-y^2} dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 e^{-r^2} r dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\frac{1}{2} e^{-r^2} \right]_0^2 d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\frac{1}{2} (e^{-4} - e^0) \right] d\theta = \left[-\frac{1}{2} (e^{-4} - e^0) \right] [\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} (1 - e^4)$$

15. Find the volume under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \le 4$.

$$V = \iint_{x^2+y^2 \le 4} \sqrt{x^2 + y^2} \, dA = \int_0^{2\pi} \int_0^2 \sqrt{r^2} \, r dr d\theta = \int_0^{2\pi} d\theta \int_0^2 r^2 dr = [\theta]_0^{2\pi} \left[\tfrac{1}{3} r^3 \right]_0^2 = 2\pi \left(\tfrac{8}{3} \right) = \tfrac{16}{3}$$

21. The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 1$ or $x^2 + y^2 = \frac{1}{2}$. So:

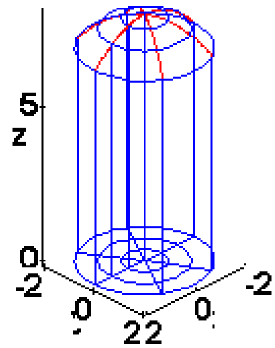
$$V = \iint_{x^2+y^2 \le \frac{1}{2}} \left(\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2} \right) dA = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} (\sqrt{1 - r^2} - r) r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{1}{\sqrt{2}}} (r\sqrt{1 - r^2} - r^2) dr = [\theta]_0^{2\pi} \left[-\tfrac{1}{3} (1 - r^2)^{\frac{3}{2}} - \tfrac{1}{3} r^3 \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \pi \left(-\tfrac{1}{3} \right) \left(\tfrac{1}{\sqrt{2}} - 1 \right) = \tfrac{\pi}{3} \left(2 - \sqrt{2} \right)$$

23. Use polar coordinates to find the volume inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$.

SOLUTION

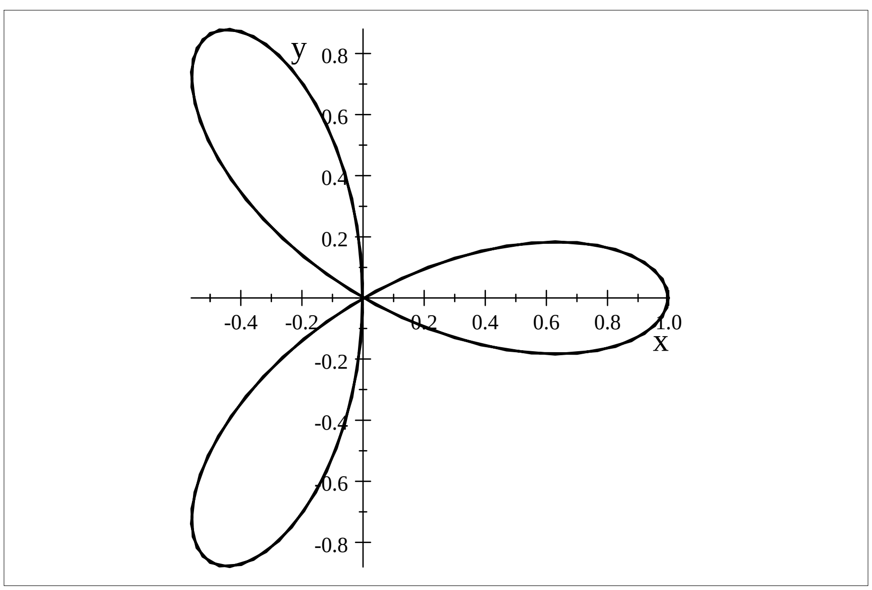


The ellipsoid intersects the x, y-plane in the circle $x^2 + y^2 = 16$. Thus, our region is bounded by the circle $x^2 + y^2 = 4$. So , in polar coordinates we have the equation $r = 2$. Next, we can solve the equation of the ellipsoid $4x^2 + 4y^2 + z^2 = 64$ for z, i.e., $z = \pm 2\sqrt{-x^2 - y^2 + 16}$ which can be rewritten in polar coordinates as $z = \pm 2\sqrt{16 - r^2}$. The volume of the solid can now be written as:

$$2\int_0^{2\pi}\int_0^2 \left(2\sqrt{16 - r^2}\right) r\,dr\,d\theta = -64\sqrt{3}\,\pi + \frac{512}{3}\pi$$

25. Use a double integral to find the area of one loop of the rose $r = \cos 3\theta$.

$r = \cos 3\theta$



$r = 0 \Rightarrow \cos 3\theta = 0$ or $3\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{6}$. Thus for the loop in the first and fourth quadrants we have $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. Using symmetry we get

$$\text{Area} = 2\int_0^{\frac{\pi}{6}}\int_0^{\cos 3\theta} r\,dr\,d\theta = \frac{\pi}{12}$$

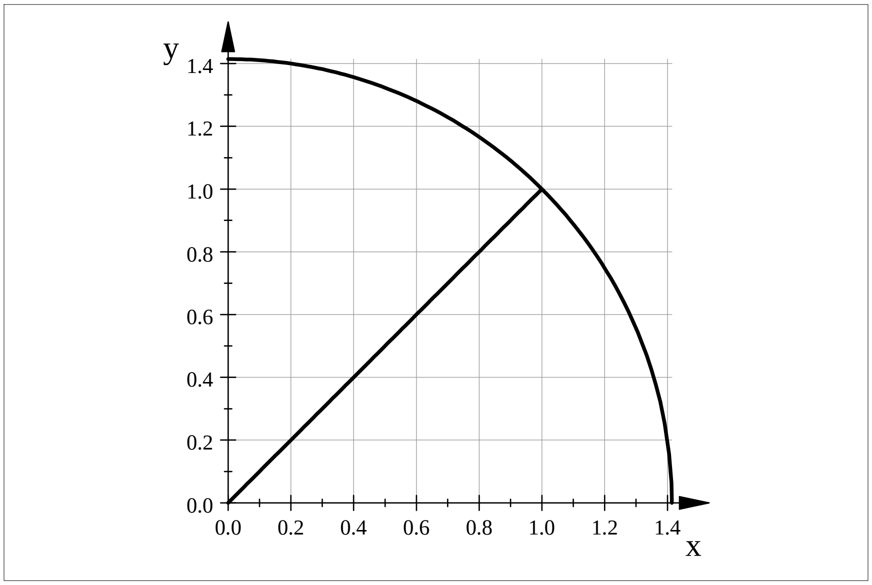
29. Evaluate

$$\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy$$

by converting to polar coordinates.

The region of integration is the part of the circle of radius $\sqrt{2}$ centered at the origin in the first quadrant below the line $y = x$.

$\sqrt{2-y^2}$



Thus

$$\int_0^1 \int_y^{\sqrt{2-x^2}} (x+y)\,dy\,dx = \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta) r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^{\sqrt{2}} (\cos\theta + \sin\theta)\,d\theta$$

$$= \frac{2\sqrt{2}}{3}[\sin\theta - \cos\theta]_0^{\frac{\pi}{4}} = \frac{2\sqrt{2}}{3}\left[\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - 0 + 1\right] = \frac{2\sqrt{2}}{3}$$

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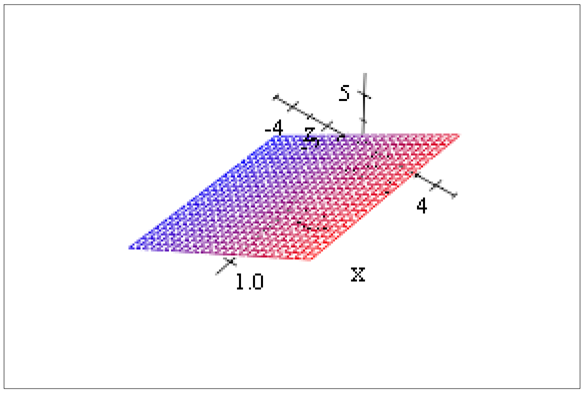
5.

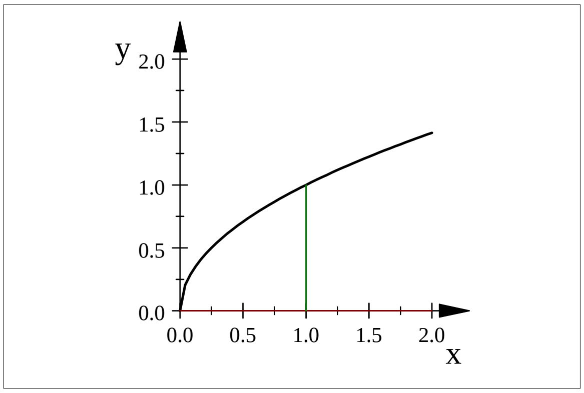
$$\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y dxdzdy = \int_0^3\int_0^1\left(\sqrt{1-z^2}\right)ze^y dzdy$$
$$= \int_0^3\left[-\frac{(1-z^2)^{\frac{3}{2}}}{3}\right]_0^1 e^y dy$$
$$= \frac{1}{3}\int_0^3 e^y dy = \frac{1}{3}(e^3-1)$$

9. Evaluate $\iiint_E 2xdV$ where $E = \{(x,y,z)|\ 0 \le z \le y, 0 \le y \le 2, 0 \le x \le \sqrt{4-y^2}\}$.

$$\iiint_E 2xdV$$
$$= \int_0^2\int_0^{\sqrt{4-y^2}}\int_0^y 2xdzdxdy = \int_0^2\int_0^{\sqrt{4-y^2}} 2xydxdy = \int_0^2(4-y^2)ydy = [2y^2 - \tfrac{1}{4}y^4]_0^2 = 4$$

11. Evaluate $\iiint_E 6xydV$ where E lies under the plane $z = 1+x+y$ and above the region in the $xy-plane$ bounded by the curves $y = \sqrt{x}, y = 0,$ and $x = 1$



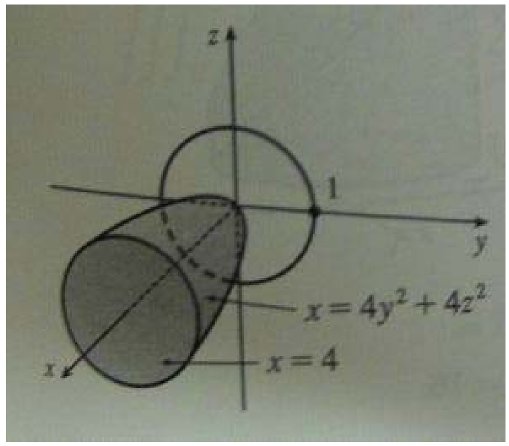


Here $E = \{(x,y,z)|0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1+x+y\}$, so

$$\iiint_E 6xydV = \int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xydzdydx = \int_0^1\int_0^{\sqrt{x}}[6xyz]_{z=0}^{z=1+x+y}dydx = \int_0^1\int_0^{\sqrt{x}} 6xy(x+y+1)dydx$$

$$= \int_0^1 [3xy^2 + 3x^2y^2 + 2xy^3]_{y=0}^{y=\sqrt{x}} \, dx = \int_0^1 \left(3x^2 + 3x^3 + 2x^{\frac{5}{2}}\right) dx = \left[x^3 + \tfrac{3}{4}x^4 + \tfrac{4}{7}x^{\frac{7}{2}}\right]_1^0 = \tfrac{65}{28}$$

17. $\iiint_E x\,dV$, where E is bounded by the parabaloid $x = 4y^2 + 4z^2$ and the plane $x = 4$



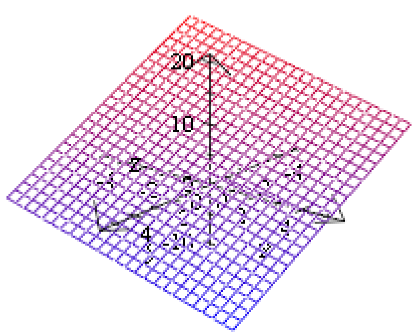
The projection on the $yz-plane$ is the disk $y^2 + z^2 \leq 1.$ Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$ we get:

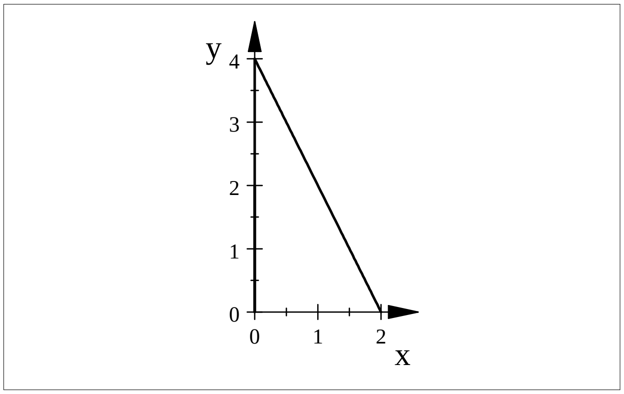
$$\iiint_E x\,dV = \iint_D \left[\int_{4y^2+4z^2}^{4} x\,dx\right] dA = \tfrac{1}{2}\iint_D \left[4^2 - (4y^2 + 4z^2)^2\right] dA = 8\int_0^{2\pi}\int_0^1 [1 - r^4]\, r\,dr\,d\theta$$

$$= 8\int_0^{2\pi} d\theta \int_0^1 [r - r^5]\,dr = 8(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \tfrac{16}{3}\pi$$

19. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $2x + y + z = 4.$

The plane $2x + y + z = 4$ intersects the xy-plane in the line $y = 4 - 2x,$





So $E = \{(x,y,z)|0 \leq x \leq 2, 0 \leq y \leq 4 - x, 0 \leq z \leq 4 - 2x - y\}$
$\Rightarrow V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dzdydx = \frac{16}{3}$