21.  
\[ f(x, y) = \ln(x + 2y) \]
\[ \nabla f(x, y) = f_x(x, y)i + f_y(x, y)j \]
\[ = \frac{1}{x + 2y}i + \frac{2}{x + 2y}j \]

23. Find the gradient vector field of \( f \)
\[ f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \]
gradient vector field = \( f_x(x, y, z)i + f_y(x, y, z)j + f_z(x, y, z)k \)
\[ = \frac{x}{\sqrt{x^2 + y^2 + z^2}}i + \frac{y}{\sqrt{x^2 + y^2 + z^2}}j + \frac{z}{\sqrt{x^2 + y^2 + z^2}}k \]
SNB check:
\[ \nabla \left( \sqrt{x^2 + y^2 + z^2} \right) = \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}}x, \frac{1}{\sqrt{x^2 + y^2 + z^2}}y, \frac{1}{\sqrt{x^2 + y^2 + z^2}}z \right) \]

25. Find the gradient vector field \( \nabla f \) of \( f \) and sketch it.
\[ f(x, y) = xy - 2x \]
Using SNB we put the cursor in the \( xy - x \), then from the compute menu we select Plot 2D, gradient to get the graph below.
\( \nabla(xy - 2x) = (y - 2, x) = (y - 2)\hat{i} + x\hat{j}. \)

Section 13.5 - p. 946 #1, 5, 13, 15, 17, 19
Find (a) the curl and (b) the divergence of the vector field.

1. \( F(x, y, z) = xyz\hat{i} - x^2y\hat{k} \)
   a. calculation of curl
   \[
   \text{curl } F = \nabla \times F = \begin{vmatrix}
   \hat{i} & \hat{j} & \hat{k} \\
   \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
   xyz & 0 & -x^2y
   \end{vmatrix} = \hat{i}(-\frac{\partial}{\partial y}x^2y - 0) - \hat{j}(\frac{\partial}{\partial x}x^2y - \frac{\partial}{\partial z}xyz) + \hat{k}(0 - \frac{\partial}{\partial y}xyz) = \]
   \(-x^2\hat{i} - j(2xy - xy) + k(0 - xz) \)
   \(-x^2\hat{i} + 3xy\hat{j} - xz\hat{k} \)
   
   SNB check: \( \nabla \times (e^x \sin y, e^x \cos y, z) = (0, 0, 0) \)

   b. divergence :
   \[
   \text{div } F = \nabla \cdot F = \frac{\partial}{\partial x} (xyz) + \frac{\partial}{\partial y} (0) + \frac{\partial}{\partial z} (-x^2y) = yz + 0 + 0 = yz
   \]
   SNB check: \( \nabla \cdot (e^x \sin y, e^x \cos y, z) = 1 \)

5. a. calculation of curl
   \[
   \text{curl } F = \nabla \times F = \begin{vmatrix}
   \hat{i} & \hat{j} & \hat{k} \\
   \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
   e^x \sin y & e^x \cos y & z
   \end{vmatrix} = \hat{i}(\frac{\partial}{\partial z} e^x \cos y) - \hat{j}(\frac{\partial}{\partial x} e^x \sin y) + \hat{k}(\frac{\partial}{\partial x} e^x \sin y) = \]
   
   SNB check: \( \nabla \times (e^x \sin y, e^x \cos y, z) = (0, 0, 0) \)

   b. divergence :
   \[
   \text{div } F = \nabla \cdot F = \frac{\partial}{\partial x} e^x \sin y + \frac{\partial}{\partial y} e^x \cos y + \frac{\partial}{\partial z} z = 1
   \]
   SNB check: \( \nabla \cdot (e^x \sin y, e^x \cos y, z) = 1 \)

13. Determine whether or not the vector field is conservative. If it is conservative find the function \( f \) such that \( F = \nabla f \)
\(
\vec{F}(x, y, z) = 2xy\hat{i} + (x^2 + 2yz)\hat{j} + y^2\hat{k}
\)

to check if the vector field is conservative, calculate the curl for the field.
and \( \vec{F} \) is defined on all of \( \mathbb{R}^3 \) with component functions which have continuous partial derivatives, so by Theorem 4, \( \vec{F} \) is conservative. Thus there exists \( f(x, y, z) \) such that \( \nabla f = \vec{F} \).

Now \( f_x = 2xy \)

so

\[
    f = x^2y + g(y, z)
\]

Hence

\[
    f_y = x^2 + g_y(y, z) = x^2 + 2yz
\]

\[
    g_y(y, z) = y^2z + h(z)
\]

Then

\[
    f_z = y^2 + h'(z) = y^2
\]

\( h(z) = k, \) where \( k \) is a constant. Thus

\[
    f(x, y, z) = x^2y + y^2z + k
\]

15. Determine whether or not the vector field is conservative. If it is conservative find the function \( f \) such that \( \vec{F} = \nabla f \).

\[
    \vec{F}(x, y, z) = ye^{-x} \vec{i} + e^{-x} \vec{j} + 2z \vec{k}
\]

The calculation of curl:

\[
    \text{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{-x} & e^{-x} & 2z \end{vmatrix} = (2y - 2y)i - (0 - 0)j + (2x - 2x)k = 0
\]

which is not equal to 0, so its not conservative.

17. Is there a vector field \( \vec{G} \) on \( \mathbb{R}^3 \) such that \( \text{curl} \vec{G} = xy^2 \vec{i} + yz^2 \vec{j} + zx^2 \vec{k} \)? Explain.

Assume there is such a \( \vec{G} \). Then

\[
    \nabla \cdot (\text{curl} \vec{G}) = \frac{\partial}{\partial x} xy^2 + \frac{\partial}{\partial y} yz^2 + \frac{\partial}{\partial z} zx^2 = y^2 + z^2 + x^2
\]

which is not equal to 0 and hence contradicts theorem 11.

19.

Calculate the curl of \( \vec{F} = f(x) \vec{i} + g(y) \vec{j} + h(z) \vec{k} \). If this is zero, then the vector field is
irrotational.

\[
\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & f(y) & f(z) \end{vmatrix} = \nabla \times (f(x), g(y), h(z)) = (0,0,0). \text{ Therefore it is irrotational.}
\]