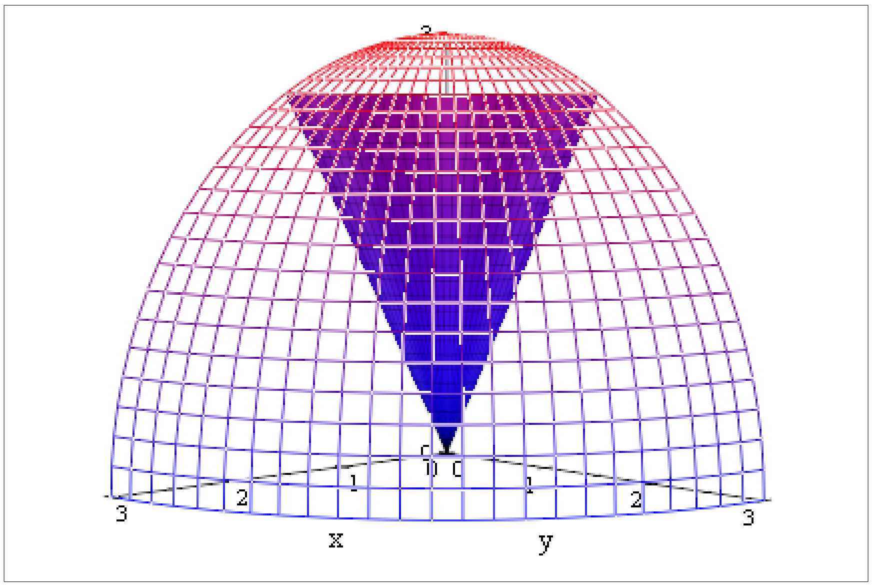
Ma 227 Homework Solutions Fall 2013 Due 9/13/2013

12.8 #3, 7, 9, 11, 15, 17, 19, 21

3) $\int_0^{\frac{\pi}{6}}\int_0^{\frac{\pi}{2}}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

The region of the integral is given by :

$E = \{(\rho,\theta,\phi)|0 \le \rho \le 3, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{6}\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\varphi = \frac{\pi}{6}$.



$$\int_0^{\frac{\pi}{6}}\int_0^{\frac{\pi}{2}}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \left(\int_0^{\frac{\pi}{6}} \sin\phi\, d\phi\right)\left(\int_0^{\frac{\pi}{2}} d\theta\right)\left(\int_0^3 d\rho\right) = [-\cos\phi]_0^{\frac{\pi}{6}}[\theta]_0^{\frac{\pi}{2}}[\tfrac{1}{3}\rho^3]_0^3$$

$$= \left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9) = \frac{9\pi}{4}(2 - \sqrt{3}).$$

$= (1 - \frac{\sqrt{3}}{2})(\frac{\pi}{2})(9) = \frac{9\pi}{4}(2 - \sqrt{3})$.

7)

Evaluate $\iiint_E \sqrt{x^2+y^2}\, dV$,where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

In cylinderical coordinate E is given by

$E = \{(r,\theta,z)|0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\, r\, dz\, dr\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^4 \sqrt{r^2}\, r\, dr\right)\left(\int_{-5}^4 dz\right)$$

$$= [\theta]_0^{2\pi}[\tfrac{1}{3}r^3]_0^4[z]_{-5}^4 = (2\pi)\left(\frac{64}{3}\right)(9) = 384\pi.$$

9) Evaluate $\int\int\int_E e^z dV$,where E is enclosed by the parabaloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$ and the $xy - plane$

In cylinderical coordinate E is bounded by the parabaloid $z = 1 + r^2$, the cylinder $r = \sqrt{5}$ and the $xy - plane$. Thus

$$\int\int\int_E e^z dV = \int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{1+r^2} e^z r dz dr d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} r[e^z]_{z=0}^{z=1+r^2} = \int_0^{2\pi}\int_0^{\sqrt{5}} r(e^{r^2+1} - 1) dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{5}} re^{r^2+1} - rdr = 2\pi\left[\frac{1}{2}e^{1+r^2} - \frac{1}{2}r^2\right]_0^{\sqrt{5}} = \pi(e^6 - e - 5)$$

11) Evaluate $\int\int\int_E x^2 dV$,where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and below the cone $z^2 = 4x^2 + 4y^2$

In cylidrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So $E = \{(r,\theta,z)|0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 0 \leq z \leq 2r\}$ and

$$\int\int\int_E x^2 dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta r dz dr d\theta = \int_0^{2\pi}\int_0^1 [r^3\cos^2\theta z]_0^{2r} dr d\theta = \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta dr d\theta$$

$$= \int_0^{2\pi}[\frac{2}{5}r^5\cos^2\theta]_0^1 d\theta = \frac{2}{5}\int_0^{2\pi}\cos^2\theta d\theta = \frac{2}{5}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{5}[\theta + \frac{1}{2}\sin\theta]_0^{2\pi} = \frac{2\pi}{5}$$

15) Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ $(a > 0)$ if S has a constant density K.

The parabaloid $z = 4x^2 + 4y^2$ intersects the plane $z = a$ when $a = 4x^2 + 4y^2$ or $x^2 + y^2 = \frac{1}{4}a$. So in cylindrical coordinates,

$E\{(r,\theta,z)|0 \leq r \leq \frac{1}{2}\sqrt{a}, 0 \leq \theta \leq 2\pi, 4r^2 \leq z \leq a\}$. Thus

$$m = \int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} K r dz dr d\theta = K\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}(ar - 4r^3) dr d\theta = K\int_0^{2\pi}\left[\frac{1}{2}ar^2 - r^4\right]_{r=0}^{r=\sqrt{a}/2} d\theta$$

$$= K\int_0^{2\pi}\frac{1}{16}a^2 d\theta = \frac{1}{8}a^2\pi K$$

Since the region is homogeneous and symmentric, $M_{yz} = M_{xz} = 0$ and

$$\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\int_{4r^2}^{a} K r z dz dr d\theta = K\int_0^{2\pi}\int_0^{\frac{\sqrt{a}}{2}}\left(\frac{1}{2}a^2 r - 8r^5\right) dr d\theta$$

$$= K\int_0^{2\pi}\left[\frac{1}{4}a^2r^2 - \frac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2} d\theta = K\int_0^{2\pi}\frac{1}{24}a^3 d\theta = \frac{1}{12}a^3\pi K$$

Hence $(x,y,z) = (0, 0, \frac{2}{3}a)$

17) In spherical coordinates, B is represented by $\{(\rho,\theta,\phi)|0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus

$$\iiint_B (x^2+y^2+z^2)^2 dV = \int_0^{2\pi}\int_0^{\pi}\int_0^5 (\rho^2)^2\rho^2 \sin\phi \, d\rho d\phi d\theta$$
$$= \int_0^{\pi} \sin\varphi d\phi \int_0^{2\pi} d\theta \int_0^5 \rho^6 d\rho = [-\cos\phi]_0^{\pi}[\theta]_0^{2\pi}[\frac{1}{7}\rho^7]_0^5$$
$$= (2)(2\pi)\left(\frac{78,125}{7}\right) = \frac{312,500}{7}\pi$$

19) In spherical coordinates ,E is represented by $\{(\rho,\theta,\phi)|1 \le \rho \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\}$

$$\iiint_E z dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^2 (\rho\cos\phi)\rho^2\sin\phi \, d\rho d\phi d\theta = \int_0^{\frac{\pi}{2}} \cos\phi \sin\varphi d\phi \int_0^{\frac{\pi}{2}} d\theta \int_1^2 \rho^3 d\rho$$
$$= [\frac{1}{2}\sin^2\phi]_0^{\frac{\pi}{2}}[\theta]_0^{\frac{\pi}{2}}[\frac{1}{4}\rho^4]_1^2 = (\frac{1}{2})(\frac{\pi}{2})(\frac{15}{4}) = \frac{15\pi}{6}.$$

21.) Evaluate $\iiint_E x^2 dV$,where E is bounded by the $xz-plane$ and the hemispheres $y = \sqrt{9-x^2-z^2}$ and $y = \sqrt{16-x^2-z^2}$

$$\iiint_E x^2 dV = \int_0^{\pi}\int_0^{\pi}\int_3^4 (\rho\sin\phi\cos\theta)^2\rho^2\sin\phi \, d\rho d\phi d\theta$$
$$= \int_0^{\pi}\cos^2\theta d\theta \int_0^{\pi}\sin^3\phi d\phi \int_3^4 \rho^4 d\rho$$
$$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi}\left[-\frac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi}\left[\frac{1}{5}\rho^5\right]_3^4$$
$$= \left(\frac{\pi}{2}\right)\left(\frac{2}{3}+\frac{2}{3}\right)\frac{1}{5}(4^5-3^5) = \frac{1562}{15}\pi$$