Matrices

The Definition of a Matrix

An $m \times n$ matrix is a rectangular array of quantities arranged in m rows and n columns. (We say the matrix is of order m by n).

Notation: Let a_{ij} $1 \leq i \leq m$ $1 \leq j \leq n$ be mn quantities. Then the matrix associated with these a_{ij}'s is denoted by

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & & . & . & . & . & a_{1n} \\ a_{21} & a_{22} & & . & . & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & . & & a_{3n} \\ . & . & . & . & . & . & & . \\ . & . & . & . & . & . & & . \\ a_{m1} & a_{m2} & & . & . & . & . & a_{mn} \end{bmatrix}$$

The quantities a_{ij} are called the elements of the matrix A.

Definition. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal $\Leftrightarrow$ they contain the same number of rows and columns and $a_{ij} = b_{ij}$ *(for all)* i,j.

Special Matrices

There are some special matrices which should be introduced.

If $n = 1 \Rightarrow A = [a_{n1}]_{m \times 1} = \begin{bmatrix} a_{11} \\ a_{12} \\ . \\ . \\ a_{m1} \end{bmatrix}$. This is called a column matrix.

If $m = 1 \Rightarrow A = [a_{1n}]_{1 \times n} = [a_{11}, a_{12}, \ldots, a_{1n}]$. This is called a row matrix.

Both column and row matrices are referred to as vectors.

When $m = n$, the we have a square matrix $\begin{bmatrix} a_{11} & . & . & a_{1n} \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & . & . & a_{nn} \end{bmatrix}$

The matrix A with every element zero is called the zero matrix. We will write $A = 0$.

The identity matrix may be defined as follows: let $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

The square matrix $I = [\delta_{ij}]_{n\times n}$ is known as the identity matrix.

$$I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & 0 & 0 \\ . & & & 0 \\ 0 & 0 & . & 1 \end{bmatrix}$$

Thus the identity matrix is the matrix with $1's$ along its diagonal and $0's$ everywhere else.

Operations on Matrices

Addition:

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then

$$A + B = [a_{ij} + b_{ij}]_{m\times n}$$

Thus $A + B$ is a matrix of order $m \times n$ whose i, j entry is $a_{ij} + b_{ij}$.

Example:

$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 6 & 4 & 7 \\ -1 & -2 & -6 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1+6 & -2+4 & 3+7 \\ 0-1 & -1-2 & 6-6 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 10 \\ -1 & -3 & 0 \end{bmatrix}$$

Example:

We can use SNB to add matrices. Thus

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 7 \\ -1 & -2 & -6 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 10 \\ -1 & -3 & 0 \end{bmatrix}$$

Subtraction:

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then $A - B$ is the matrix defined by

$$A - B = [a_{ij} - b_{ij}]_{m\times n}$$

Note: One can add and subtract only matrices of the same order. Such matrices are called conformable.

Scalar Multiplication

Let k be a scalar and A a matrix of real numbers of order $m \times n$. Then

$$kA = [k \cdot a_{ij}]_{m\times n}$$

Example:

$$5\begin{bmatrix} -1 & 0 & 5 & 7 \\ 2 & -8 & 4 & 22 \\ -7 & 1 & 0 & 6 \\ 8 & 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 25 & 35 \\ 10 & -40 & 20 & 110 \\ -35 & 5 & 0 & 30 \\ 40 & 15 & -15 & 20 \end{bmatrix}$$

Some Properties of Addition and Scalar Multiplication

We now list the basic properties of vector addition and scalar multiplication.

Theorem

Let A, B and C be conformable $m \times n$ matrices whose entries are real numbers, and k and p arbitrary scalars. Then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$
3. There is an $m \times n$ matrix 0 such that $0 + A = A$ for each A.
4. For each A there is an $m \times n$ matrix $-A$ such that $A + (-A) = 0$.
5. $k(A + B) = kA + kB$
6. $(k + p)A = kA + pA$
7. $(kp)A = k(pA)$.

Proof:

(1) $A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$

$= [b_{ij} + a_{ij}]_{m\times n}$ commutativity of real numbers.

$= B + A$

(4) Note that $(-1)A = [-a_{ij}]_{m\times n}$ $\Rightarrow A + (-1)A = 0_{m\times n}$

Remark: We denote $(-1)A$ by $-A$.

The Transpose of a Matrix

If A is an $m \times n$ matrix, the transpose of A, denoted A^T, is the $n \times m$ matrix whose entry a_{st} is the same as the entry a_{ts} in the matrix A. Thus one gets the transpose of A by interchanging the rows and the columns of A.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -2 \\ 4 & 10 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 10 \\ -1 & -2 & 9 \end{bmatrix}$$

We note the following:

- $(A^T)^T = A$.
- $(A + B)^T = A^T + B^T$.
- For any scalar r, $(rA)^T = rA^T$.
- If A is a diagonal matrix, then $A = A^T$.
- A square matrix is said to be symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$.

Example:

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 0 \end{bmatrix}$ is symmetric and $\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is skew-symmetric.