Properties of Matrices

Multiplication of Matrices

Consider a system of m equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \cdots &= \cdots \\ \cdots &= \cdots \\ \ddots &= \ddots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m \end{aligned}$$

Here the a_{ij} and d_i are given scalars and the x_j are the unknowns.

If we let $A = [a_{ij}]_{m\times n}$,

$$X = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad \text{and} \quad D = \begin{bmatrix} d_1 \\ \vdots \\ \vdots \\ d_m \end{bmatrix}_{m\times 1}$$

Then it is natural to write $AX = D$ to represent the system above. Hence we want

$$\begin{bmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_m \end{bmatrix}$$

to be the same as our system above. $\Rightarrow$ that multiplication should be defined by

$$d_i = \sum_{j=1}^{n} a_{ij}x_j$$

Notice that A is $m \times n$ and X is $n \times 1$ and D is $m \times 1$. Thus to multiply two matrices we must have the number of columns of the first matrix equal to the number of rows of the second matrix. To multiply two matrices A and B together, where B is not a column matrix, we extrapolate as follows:

$$\begin{bmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{11} & \cdots & \cdots & \cdots & b_{1p} \\ b_{21} & \cdots & \cdots & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & \cdots & b_{np} \end{bmatrix}_{n\times p} = \begin{bmatrix} c_{11} & \cdots & \cdots & \cdots & c_{1p} \\ c_{21} & \cdots & \cdots & \cdots & c_{2p} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} & \cdots & \cdots & \cdots & c_{mp} \end{bmatrix}_{m\times p}$$

We see that

$$c_{11} = \sum_{k=1}^{n} a_{1k} b_{k1}$$

and in general that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Definition. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be matrices. Then $A\,B$ is the $m \times p$ matrix $C,$ where

$$C = [c_{ij}]_{m\times p} = \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]_{m\times p}$$

Remark. $A\,B \neq B\,A$ necessarily.

In fact $B\,A$ need not be defined. For example if A is 2×3 and B is 3×4, then $A\,B$ will be 2×4 whereas $B\,A$ is not defined.

Example:

$$\begin{bmatrix} 1 & -1 & 0 \\ 4 & 1 & -1 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 3 & 4 \\ -1 & -5 \\ 1 & 2 \end{bmatrix}_{3\times 2} = \begin{bmatrix} (1)(3)+(-1)(-1)+(0)(1) & (1)(4)+(-1)(-5)+(0)(2) \\ (4)(3)+(1)(-1)+(-1)(1) & (4)(4)+(1)(-5)+(-1)(2) \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 4 & 9 \\ 10 & 9 \end{bmatrix}_{2\times 2}$$

$$\begin{bmatrix} 3 & 4 \\ -1 & -5 \\ 1 & 2 \end{bmatrix}_{3\times 2} \times \begin{bmatrix} 1 & -1 & 0 \\ 4 & 1 & -1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 19 & 1 & -4 \\ -21 & -4 & 5 \\ 9 & 1 & -2 \end{bmatrix}_{3\times 3}$$

Note that using Evaluate in SNB gives the same result.

$$\begin{bmatrix} 3 & 4 \\ -1 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 4 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 19 & 1 & -4 \\ -21 & -4 & 5 \\ 9 & 1 & -2 \end{bmatrix}$$

The following occur often for matrices.

1. $A\,B \neq B\,A$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. $A\,B = 0$ but neither $A = 0$ or $B = 0$

Example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. $A\,B = A\,C$ but $B \neq C$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem

Assume that k is an arbitrary scalar, and that $A,\ B,\ C$ and I are matrices of sizes such that the indicated operations can be performed. Then

1. $IA = A, \qquad BI = B$

2. $A(BC) = (AB)C$

3. $A(B+C) = AB+AC, \qquad A(B-C) = AB-AC$

4. $(B+C)A = BA+CA, \qquad (B-C)A = BA-CA$

5. $k(AB) = (kA)B = A(kB)$

6. $(AB)^T = B^T A^T.$

Proof of 6

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$. Then $A^T = \left[a'_{ij}\right]_{n\times m}$ and $B^T = \left[b'_{ij}\right]_{p\times n}$, where $a'_{ij} = a_{ji}$ and $b'_{ij} = b_{ji}$. The (i,j) entry of $B^T A^T$ is

$$\sum_{k=1}^{n} b'_{ik}a'_{kj} = \sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}b_{ki}$$

This last term is the (j, i) entry of AB which means that it is the (i, j) entry of $(AB)^T$.

Operations with the Identity Matrix

Consider the 3×3 identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the matrix

$$A = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 3 & 0 & 1 & 6 \\ 5 & 7 & 9 & 8 \end{bmatrix}$$

Note the following:

$$IA = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 3 & 0 & 1 & 6 \\ 5 & 7 & 9 & 8 \end{bmatrix} = A$$

Also,

$$AI = A$$

Cramer's Rule

Cramer's Rule: Let A be an $n \times n$ matrix, $A = [a_{ij}]_{n \times n}$ and denote by $A_{(j)}$ the $n \times n$ matrix formed by replacing the elements a_{ij} of the jth column of A by the numbers $k_i,\ i = 1, \ldots.., n$. If $|A| \neq 0$, the system of n linear equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= k_2 \\ \vdots &= \vdots \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= k_n \end{aligned}$$

has the unique solution

$$x_1 = \frac{\det A_{(1)}}{\det A} \qquad x_2 = \frac{\det A_{(2)}}{\det A}, \ldots x_n = \frac{\det A_{(n)}}{\det A}$$

Example. Solve

$$\begin{aligned} x + 3y - 2z &= 1 \\ 4x - 2y + z &= -15 \\ 3x + 4y - z &= 3 \end{aligned}$$

by Cramer's Rule

$$\det A = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -25$$

$$x = \frac{\begin{vmatrix} 1 & 3 & -2 \\ -15 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{-25} = -\frac{14}{5}, \qquad y = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 4 & -15 & 1 \\ 3 & 3 & -1 \end{vmatrix}}{-25} = \frac{19}{5}, \qquad z = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 4 & -1 & -15 \\ 3 & 4 & 3 \end{vmatrix}}{-25} = \frac{19}{5}$$

Systems of Equations: Elimination Using Matrices

Remark: We cannot use Cramer's Rule if the determinant of the coefficients in a system of n equations in n unknowns equals 0 or if the number of equations is not equal to the number of unknowns.

Example:

Solve the system

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2 \\ 4x_1 + 5x_2 + 2x_3 &= 7 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 2 \\ 4x_1 + 5x_2 + 2x_3 &= 7 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 2 & 0 & 7 \end{bmatrix}$$

The matrix on the right that we have associated with the given system is called the augmented matrix of the system. The matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 2 & 0 \end{bmatrix}$$

is called the coefficient matrix of the system, and $C = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$ is called the constant matrix (vector) of the system. It is clear that we can rewrite our system as

$$AX = C$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

$$\begin{array}{c} x_1 + x_2 + 2x_3 + x_4 = 5 \\ 0 + x_2 - 5x_3 - 4x_4 = -8 \\ 0 + x_2 - 6x_3 - 4x_4 = -13 \end{array} \leftrightarrow \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -6 & -4 & -13 \end{bmatrix}$$

$$\begin{array}{c} x_1 + 0x_2 + 7x_3 + 5x_4 = 13 \\ 0 + x_2 - 5x_3 - 4x_4 = -8 \\ 0 + 0 - x_3 = -5 \end{array} \leftrightarrow \begin{bmatrix} 1 & 0 & 7 & 5 & 13 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & -1 & 0 & -5 \end{bmatrix}$$

$$\begin{array}{c} x_1 + 0x_2 + 0x_3 + 5x_4 = -22 \\ 0 + x_2 + 0x_3 - 4x_4 = 17 \\ 0 + 0 + x_3 = 5 \end{array} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 5 & -22 \\ 0 & 1 & 0 & -4 & 17 \\ 0 & 0 & 1 & 0 & 5 \end{bmatrix}$$

Thus $x_3 = 5, \quad x_1 + 5x_4 = -22, \quad x_2 - 4x_4 = 17$
or $x_3 = 5, \quad x_4 = t \quad x_1 = -5t - 22 \quad x_2 = 4t + 17$

Note that we have an infinite number of solutions. However, if our operations had led to

$$\begin{bmatrix} 1 & 0 & 0 & 5 & -22 \\ 0 & 1 & 0 & -4 & 17 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix},$$

then there would not be any solution to the system, since the last row of the matrix would imply

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 5$$

which is clearly impossible.

Definition: Systems of linear equations that have no solution are called inconsistent systems; systems that have at least one solution are said to be consistent.

Elementary Row Operations On Matrices I

Equivalent Systems

Two linear systems are **equivalent** if they have the same solutions.

Three Elementary Operations

Three basic elementary operations are used to transform systems to equivalent systems. These are:

1. Interchanging the order of the equations in the system.

2. Multiplying any equation by a nonzero constant.

3. Replacing any equation in the system by its sum with a nonzero constant multiple of any other equation in the system (elimination step).

Theorem:

Suppose that an elementary row operation is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

Operating on the rows of a matrix is equivalent to operating on equations. The row operations that are allowed are the same as the row operations on linear systems of equations:

1. Interchanging the rows.

2. Multiplying any row by a nonzero constant.

3. Replacing any row by its sum with a nonzero constant multiple of any other row. (Add a multiple of one row to a different row.)

Example:

Find all solutions to the following system of equations

$$\begin{aligned} 3x + 4y + z &= 1 \\ 2x + 3y \quad\; &= 0 \\ 4x + 3y - z &= -2 \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{bmatrix}$$

We could get a 1 in the first row and first column by multiplying row 1 (R_1) by $\frac{1}{3}$. However, we can get a 1 in this spot without obtaining fractions by subtracting row 2 from row 1. Doing this we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{bmatrix}$$

We now use the 1 that we have in the first row and first column to get zeroes below it. Hence we multiply row 1 by -2 and add it to row 2. We also multiply row 1 by -4 and add it to row 3. we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -5 & -6 \end{bmatrix}$$

We now use the 1 in the second row and second column to get zeroes above and below it. We get

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \end{bmatrix}$$

We may now get a 1 in the third row and third column by multiplying the third row by $-\frac{1}{7}$. Doing this yields

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & \frac{8}{7} \end{bmatrix}$$

Now we can use the one in the last row to get zeroes for the entries above it. Doing this we get

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{8}{7} \end{bmatrix}$$

Clearly the solution is $x = -\frac{3}{7}, y = \frac{2}{7}, z = \frac{8}{7}$.

Gaussian Elimination

Definition: A matrix is said to be in row-echelon form (and will be called a row-echelon matrix) if it satisfies the following three conditions:

1. All zero rows (consisting entirely of zeroes) are at the bottom.

2. The first nonzero entry from the left in each nonzero row is a 1, called the leading 1 for that row.

3. Each leading 1 is to the right of all leading $1's$ in the rows above it.

Definition: A row-echelon matrix is said to be in reduced row-echelon form (and will be called a reduced row-echelon matrix) if it satisfies the following condition:

4. Each leading 1 is the only nonzero entry in its column.

Examples

The matrices

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 9 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are all in echelon form. The third is in reduced row-echelon form.

Example:

None of the matrices below is in row-echelon form.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ violates (1)} \qquad \begin{bmatrix} 1 & 0 & 6 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ violates (2)}$$

$$\begin{bmatrix} 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & 5 & -2 \end{bmatrix} \text{ violates (3)}$$

Example:

Insert the appropriate (elementary) matrices corresponding to the indicated row operations that transform

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix}$$

to row-reduced echelon form.

Adding $2 \times R_1$ to R_4 yields

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

interchanging R_2 and R_3

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

Adding $(-1) \times R_3$ to R_4 yields

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adding $2 \times R_1$ to R_2 yields

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & -1 & 0 & 4 & -8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Multiplying R_1 and R_2 by -1 yields

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & -1 & 0 & 4 & -8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Eliminating the 1 in row one column 2 yields

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example:

Solve the system $AX = C$, where

$$A = \begin{bmatrix} -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & -7 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } C = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_1 - x_2 + 2x_4 \\ x_3 - 3x_4 \\ 2x_1 + x_2 \\ 2x_1 + 2x_2 + x_3 - 7x_4 \end{bmatrix}$$

Hence the system is

$$\begin{aligned} -x_1 - x_2 + 2x_4 &= -4 \\ x_3 - 3x_4 &= 0 \\ 2x_1 + x_2 &= 0 \\ 2x_1 + 2x_2 + x_3 - 7x_4 &= 8 \end{aligned}$$

Solution is: $\{x_3 = 3x_4, x_1 = -2x_4 - 4, x_2 = 4x_4 + 8, x_4 = x_4\}$. We see that this is indeed the case from the row-reduced echelon form

$$\begin{bmatrix} -1 & -1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & -7 & 8 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & 0 & 2 & -4 \\ 0 & 1 & 0 & -4 & 8 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Remark: In general, when the augmented matrix of a system has been carried to (reduced) row-echelon form, variables containing a leading 1 are called leading variables. The non-leading variables (if any) end up as parameters in the final solution, and the leading variables are given (by the equations) in terms of these parameters.

Theorem

Every matrix can be brought to (reduced) row-echelon form by a series of elementary row operations.

Gauss Elimination (Gauss Algorithm)

Step 1. If the matrix consists entirely of zeros, stop, - it is already in row-echelon form.

Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it c), and move the row containing that entry to the top position.

Step 3. Now multiply that row by $\frac{1}{c}$ to create a leading 1.

Step 4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

Step 5. Cover the top row and repeat steps 1-4 on the submatrix consisting of the remaining rows.

The process stops when either no rows remain at step 5 or the remaining rows consist of zeros.

This procedure produces a matrix that is in row-echelon form. To get a matrix that is in row-reduced echelon form we add one more step.

Step 6. Starting with the last nonzero row work upward: For each row introduce zeros above the leading 1 by adding suitable multiples to the corresponding rows.

Note: The entire procedure (steps1-6) is often called Gauss-Jordan Elimination.

Example:

Solve the system

$$\begin{aligned} x_1 - 3x_2 + x_3 - x_4 &= -1 \\ -x_1 + 3x_2 + 3x_4 + x_5 &= 3 \\ 2x_1 - 6x_2 + 3x_3 - x_5 &= 2 \\ -x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 6 \end{aligned}$$

using the Gauss algorithm and back substitution.
The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ -1 & 3 & 0 & 3 & 1 & 3 \\ 2 & -6 & 3 & 0 & -1 & 2 \\ -1 & 3 & 1 & 5 & 1 & 6 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & -3 & 0 & -3 & 0 & -4 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding equations are

$$\begin{aligned} x_1 - 3x_2 - 3x_4 &= -4 \\ x_3 + 2x_4 &= 3 \\ x_5 &= -1 \end{aligned}$$

Thus

$$x_5 = -1, \qquad x_3 = 3 - 2x_4, \qquad x_1 = 3x_2 + 3x_4 - 4$$

Letting $x_2 = s$ and $x_4 = t$ we have the infinite set of solutions

$$x_1 = 3s + 3t - 4, \qquad x_3 = 3 - 2t, \qquad x_5 = -1$$

Inverse of a Matrix

Definition: If A is a square $n \times n$ matrix, a matrix B is called the inverse of A if and only if

$$AB = I \quad \text{and} \quad BA = I.$$

A matrix A that has an inverse is called an invertible or nonsingular matrix.

Example

Show that the matrix $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

$BA = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so B is indeed an inverse of A.

Example

The matrix $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ is not invertible. For if $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ is any 2×2 matrix, then

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b_{11}+b_{21} & b_{12}+b_{22} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Theorem
If B and C are both inverses of $A,$ then $B = C.$

Proof:
Since B and C are both inverses of $A,$ $CA = I = AB.$ Hence

$$B = IB = (CA)B = C(AB) = CI = C$$

Remark: If A is invertible then the (unique) inverse of A is denoted by A^{-1}.

Example:
Under what conditions is the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible. When A is invertible, find A^{-1}.

We seek a matrix $A^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We could form two augmented matrices (one for each system) and then put each of them in reduced row-echelon form. However, we may just as well do the entire reduction at the same time. Thus

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}, \text{ row echelon form: } \begin{bmatrix} 1 & 0 & \frac{d}{da-cb} & -\frac{b}{da-cb} \\ 0 & 1 & -\frac{c}{da-cb} & \frac{1}{da-cb}a \end{bmatrix}.$$

Thus we see that A is invertible $\Leftrightarrow ad - bc \neq 0.$ If this condition holds, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem
Suppose a system of n equations in n variables is written in matrix form as

$$AX = B$$

If the $n \times n$ coefficient matrix A is invertible, then the system has the unique solution

$$X = A^{-1}B$$

Corollary

Suppose the system $AX = 0$ of n equations in n unknowns has a nontrivial solution. Then A cannot be invertible.

Example:

Let $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$. Find A^{-1} and use it to solve the system of equations $AX = B$.

We use Maple to find A^{-1}.

$\begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, inverse: $\begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix} = A^{-1}$ Then

$$X = A^{-1}B = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ -9 \\ -13 \end{bmatrix}$$

The Calculation of A^{-1} by Gauss-Jordan Elimination

Suppose we want to find the inverse of

$$A = \begin{bmatrix} a_{11} & . & . & . & a_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{bmatrix}$$

Then we want a matrix B such that $AB = I$. If b_{i1} are the elements in the first column of B then

$$A \begin{bmatrix} b_{11} \\ . \\ . \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ . \\ 0 \end{bmatrix} \Rightarrow \text{we must solve } AX = \begin{bmatrix} 1 \\ 0 \\ . \\ 0 \end{bmatrix}.$$

We can solve this system by forming $\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \\ . \\ 0 \end{matrix} \end{array}\right]$, and then row reducing this to reduced row-echelon form.

If b_{i2} are elements in the second column of $B \Rightarrow$

$$A\begin{bmatrix} b_{12} \\ . \\ . \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ . \\ 0 \end{bmatrix} \Rightarrow \text{ we must solve } AX = \begin{bmatrix} 0 \\ 1 \\ . \\ 0 \end{bmatrix}\text{. Therefore form}$$

$$\left[\begin{array}{cc} & 0 \\ A & 1 \\ & . \\ & 0 \end{array}\right]$$ and reduce to reduced row-echelon form.

In general we need to solve the n systems

$$AX = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} = C_j$$

where C_j is the *jth* column of the $n \times n$ identity matrix. Rather doing the same row reduction n times, we use the Gauss-Jordan method which computes A^{-1} by solving all n systems at the same time.

We can solve all these systems at once by forming $[A|I]$ and then putting this matrix in reduced row-echelon form.

Remark. The system will have a unique solution $\Leftrightarrow \det A \neq 0 \Leftrightarrow$ we can use Cramer's rule.
Theorem. A is nonsingular $\Leftrightarrow$ A is invertible $\Leftrightarrow$ A has rank n $\Leftrightarrow$ A can be row reduced to the identity matrix.

Example. Find A^{-1} for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Note that $\det A = 5 \neq 0$

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & +\frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

Slide Show Example:

To view this example you will need Real Player G2 installed on your machine. To get is hold down the Ctrl key and click on Real. Now hold down the Ctrl key and click Inverse Slide Show to see a slide show that explains how you find the inverse of a matrix. You may also view the tex file from which this slide show was made example by holding down the Ctrl key and clicking Inverse File .

Example:

Find A^{-1} for $A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. We form $\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \to_{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \to_{-2R_1+R_2 \quad -R_1+R_3} \begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix} \to_{-R_2+R_3 \quad 4R_2+R_1} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix} \to_{-R_2} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix} \to_{\left(-\frac{1}{2}\right)R_3} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \to_{-11R_3+R_1 \quad 3R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Check: $AA^{-1} = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

or using SNB $\begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$, inverse: $\begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$