Ma 227 Review of Gradient, Curl, Divergence, and Line Integrals

Vector Fields, Gradient, Divergence, Curl

Definition Let D denote a subset of the plane. A **vector field** on D is a function $\mathbf{F}$ that assigns to each point (x,y) in D a two-dimensional vector $\mathbf{F}(x,y)$. In terms of its component functions, the vector field $\mathbf{F}$ is given by

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = \langle P(x,y), Q(x,y)\rangle$$

or, for short,

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$$

- Definition Let D denote a subset of space. A **vector field** on D is a function $\mathbf{F}$ that assigns to each point (x,y,z) in D a three-dimensional vector $\mathbf{F}(x,y,z)$. In terms of its component functions, the vector field $\mathbf{F}$ is given by

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

or, for short,

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

Definition If f is a scalar function of two variables, its **gradient vector field** ∇f is defined by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

If f is a scalar function of three variables, its **gradient vector field** ∇f is defined by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Definition $\mathbf{F}$ is a **conservative vector field** if there exists a scalar function f such that $\nabla f = \mathbf{F}$. In this case f is called a **potential function** for $\mathbf{F}$.

Example:

Let $\Phi(x,y,z) = xyz + 3x^4y^2z^3$. Then $\nabla\Phi = (yz + 12x^3y^2z^3)\vec{i} + (xz + 6x^4yz^3)\vec{j} + (xy + 9x^4y^2z^2)$

Divergence

A vector field is a vector-valued function. If

$$\vec{F}(x,y,z) = [p(x,y,z), q(x,y,z), r(x,y,z)] = p\vec{i} + q\vec{j} + r\vec{k}$$

is a vector field, then the scalar

$$\nabla \cdot \vec{F} = \operatorname{div}\vec{F} = \frac{\partial p}{\partial x}(a,b,c) + \frac{\partial q}{\partial y}(a,b,c) + \frac{\partial r}{\partial z}(a,b,c)$$

is the divergence of F at the point (a,b,c).

Curl

If $\vec{F}(x,y,z) = (p(x,y,z), q(x,y,z), r(x,y,z)) = p\vec{i} + q\vec{j} + r\vec{k}$ is a vector field, then the vector

$$\nabla \times \vec{F} = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}, \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}, \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) = \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\vec{i} + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\vec{j} + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\vec{k}$$

is called the curl of F.

Example:

Let $\vec{F} = xy\vec{i} + xz^2\vec{j} + ze^x \sin y\vec{k}$. Find $\operatorname{div}\vec{F}$ and $\operatorname{curl}\vec{F}$.

$$\nabla \bullet \vec{F} = \operatorname{div}\vec{F} = y + e^x \sin y$$

$$\nabla \times \vec{F} = \operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz^2 & ze^x \sin y \end{vmatrix}$$

$$= (ze^x \cos y - 2xz)\vec{i} - ze^x \sin y\vec{j} + (z^2 - x)\vec{k}$$

Line Integrals

We define the line integral as follows:

A curve C may be described in three dimensions via

$$x = f(t); \qquad y = g(t); \qquad z = h(t) \quad a \le t \le b$$

or

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \quad a \le t \le b$$

If

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

then

$$\int_C \vec{F} \bullet d\vec{r} = \int_C Pdx + Qdy + Rdz = \int_a^b \vec{F}(f(t), g(t), h(t)) \bullet \vec{r}\,'(t)dt$$

$$= \int_a^b \{P(f(t), g(t), h(t))f'(t) + Q(f(t), g(t), h(t))g'(t) + R(f(t), g(t), h(t))h'(t)\}dt$$

Example:

Evaluate $\int_C \vec{F} \bullet \vec{dr}$, where $\vec{F} = (x + 2y)\vec{i} + (x^2 - y^2)\vec{j}$ and C is the line segment joining $(0,0)$ and $(1,1)$.

SOLUTION

We have to parametrize $\vec{F}$ and $\vec{r}$ first. Since we are moving from 0 to 1 along the line $x = y$, it behoves us to set $x = y = t$ as our parameter.

Then

$$\vec{F} = (t + 2t)\vec{i} + (t^2 - t^2)\vec{j} = 3t\vec{i}.$$

Next,

$$\vec{r} = x(t)\vec{i} + y(t)\vec{j} = t\vec{i} + t\vec{j}.$$

$$\vec{r}\,'(t) = \vec{i} + \vec{j}.$$

Finally,

$$\vec{F} \bullet \vec{r}\,'(t) = 3t\vec{i} \cdot (\vec{i} + \vec{j}) = 3t$$

Integrate now:

$$\int_C \vec{F} \bullet \overrightarrow{dr} = \int_0^1 3t \; dt = \frac{3}{2}.$$

Path Independence

Certain line integrals depend only on the integrand and endpoints *A* and *B*. Such integrals are called path independent or are said to be independent of the path.

Often one must consider situations in which the path *C* is a closed curve. Hence the starting point *A* and ending point *B* are the same. This is usually written as

$$\oint_C \vec{F} \cdot d\vec{r}.$$

For plane curves we take the positive direction of *C* so that the interior of the closed curve is always to the left as *C* is traversed.

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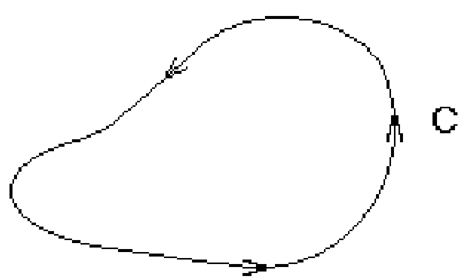


The following are equivalent:

$\int_C \vec{F} \cdot d\vec{r}$ is path independent $\leftrightarrow$ there exists a G such that $\vec{F} = \nabla G$

$\leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed path C

$\leftrightarrow \nabla \times \vec{F} = curl\vec{F} = 0$

Note in two dimensions with $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ the last condition becomes

$$Q_x = P_y$$

Example:

Let $\vec{F} = y^2z^3\vec{i} + 2xyz^3\vec{j} + (3xy^2z^2 + z)\vec{k}$. Show that curl $\vec{F} = \vec{0}$. Evaluate $\oint_C \vec{F} \bullet \overrightarrow{dr}$, where C is the ellipse in x, y, z -space given by

$$\frac{x^2}{4} + \frac{y^2}{9} = 1, z = 3.$$

SOLUTION

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 + z \end{bmatrix} = 6xyz^2\vec{i} + 2yz^3\vec{k} + 3y^2z^2\vec{j} - 2yz^3\vec{k} - 6xyz^2\vec{i} - 3y^2z^2\vec{j} = \vec{0}.$$

We have shown that for the given $\vec{F}$, $\text{curl}\vec{F} = 0$. Therefore the integral is path-independent, so the integral around a closed curve is zero. Thus $\oint_C \vec{F} \bullet \vec{dr} = 0$.

Example:

Consider the $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{F} = (2xyz + z^2y)\vec{i} + (x^2z + z^2x)\vec{j} + (x^2y + 2xyz)\vec{k}$. Show that $\nabla \times \vec{F} = \vec{0}$. What does this tell you about $\oint_C \vec{F} \cdot \vec{dr}$, where C is any closed curve?

SOLUTION

$$\nabla \times \vec{F} = curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz + z^2y & x^2z + z^2x & x^2y + 2xyz \end{vmatrix}$$

$$= (x^2 + 2xz - x^2 - 2zx)\vec{i} - (2xy + 2yz - 2xy - 2zy)\vec{j} + (2xz + z^2 - 2xz - z^2)\vec{k}$$

$$= \vec{0}$$

Then $\oint_C \vec{F} \cdot \vec{dr} = 0$ for any closed curve C. Or, equivalently, $\int_C \vec{F} \cdot \vec{dr}$ is independent of the path taken between two given points.

Example

Find a function $\Phi(x,y,z)$ such that $\nabla\Phi = \vec{F}$, where $\vec{F}$ is the vector field above.

SOLUTION

$\vec{F} = \nabla\Phi = \frac{\partial\Phi}{\partial x}\vec{i} + \frac{\partial\Phi}{\partial y}\vec{j} + \frac{\partial\Phi}{\partial z}\vec{k}$ We set equal the corresponding components.

$\frac{\partial\Phi}{\partial x} = 2xyz + z^2y \qquad \Rightarrow$

$$\Phi(x,y,z) = x^2yz + xyz^2 + h(y,z)$$

$$\frac{\partial\Phi}{\partial y} = x^2z + z^2x + h_y = x^2z + z^2x$$

Thus $h = g(z)$

$$\frac{\partial \Phi}{\partial z} = x^2y + 2xyz + g'(z) = x^2y + 2xyz$$

$g(z) = C,$ a constant

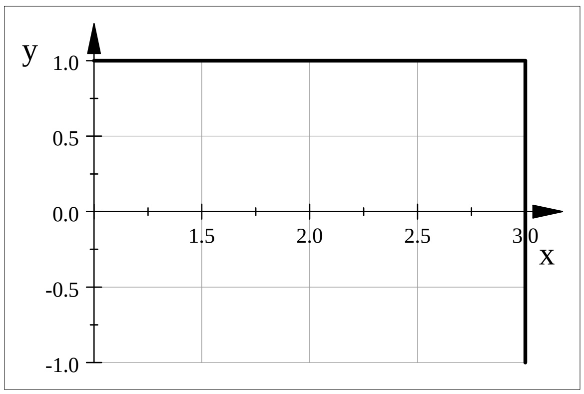
Finally, we have

$$\Phi(x,y,z) = x^2yz + xyz^2 + C$$

Example:
Evaluate

$$\int_C (x+2y)dx + (x^2 - y^3)dy$$

where C consists of the segments from $(1,1)$ to $(3,1)$ and $(3,1)$ to $(3,-1)$. Sketch C.
Solution: The path C is shown below.



Let C_1 be the segment from $(1,1)$ to $(3,1)$ and C_2 be the segment from $(3,1)$ to $(3,-1)$. Then $C = C_1 \cup C_2$.

$$\begin{aligned}\int_C (x+2y)dx + (x^2-y^3)dy &= \int_{C_1} (x+2y)dx + (x^2-y^3)dy + \int_{C_2} (x+2y)dx + (x^2-y^3)dy \\ &= \int_{C_1} (x+2(1))dx + \int_{C_2} \left((3)^2 - y^3\right)dy \\ &= \int_1^3 (x+2)dx + \int_1^{-1} (9-y^3)dy = 8 - 18 = -10\end{aligned}$$

Green's Theorem

Theorem: Let $P(x,y)$ and $Q(x,y)$ be functions of two variables which are continuous and have continuous first partial derivatives in some rectangular region H in the $x,y-$ plane. If C is a simple, closed, piecewise smooth curve lying entirely in H, and if R is the bounded region enclosed by C, then

$$\oint_C \{P(x,y)dx + Q(x,y)dy\} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

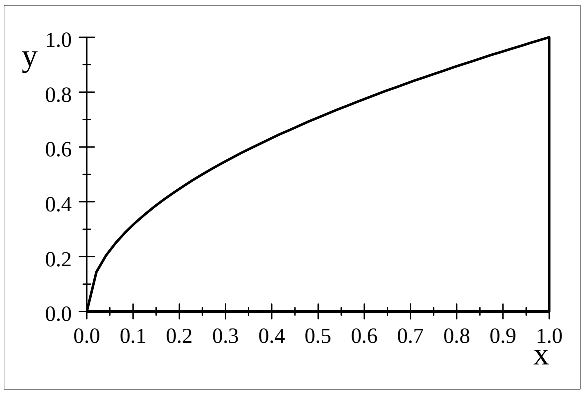
Note: Green's Theorem applies only to a closed curve.
Corollary: Let R be a bounded region in the $x,y-$ plane. Then the area of R is given by

$$A = \frac{1}{2}\oint_C (xdy - ydx) = \oint_C xdy = -\oint_C ydx$$

Example:
Evaluate

$$\oint_C (1 + \tan x)dx + (x^2 + e^y)dy$$

Where C is the positively oriented boundary of the region R enclosed by the curves $y = \sqrt{x}$, $x = 1$, and $y = 0$. Be sure to sketch C.
Solution:
The region enclosed by C is shown below.



We use Green's Theorem to evaluate the integral since C is a closed curve.

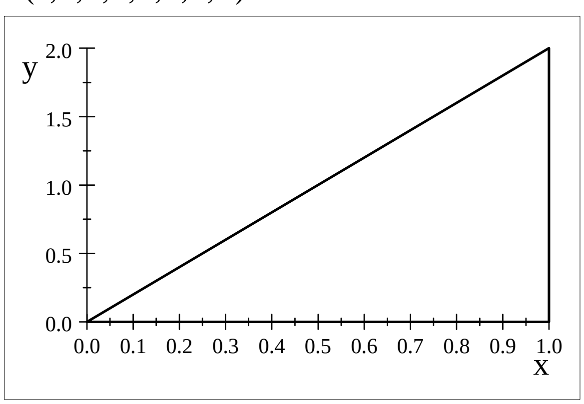
$$\oint_C (1 + \tan x)dx + (x^2 + e^y)dy = \int\int_R \left(\frac{\partial(x^2 + e^y)}{\partial x} - \frac{\partial(1 + \tan x)}{\partial y} \right) dA$$

$$= \int_0^1 \int_0^{\sqrt{x}} (2x - 0)dydx = 2\int_0^1 x^{\frac{3}{2}} dx = \frac{4}{5}$$

Example:
Verify that Green's Theorem is true for the line integral

$$\oint_C xydx + x^2 dy$$

where C is the triangle with vertices $(0,0), (1,0),$ and $(1,2)$.
Solution:
The triangle is shown below. $(0,0,1,0,1,2,0,0)$



The boundary consists of 3 line segments $C_1 : 0 \le x \le 1, y = 0; C_2 : 0 \le y \le 2, x = 1; C_3 : y = 2x$, $x = 1$ to 0.

Thus

$$\oint_C xydx + x^2dy = \int_0^1 0dx + \int_0^2 (1)dy + \int_1^0 (2x^2 + 2x^2)dx = \frac{2}{3}$$

Also

$$\begin{aligned}\oint_C xydx + x^2dy &= \iint_R \left(\frac{\partial(x^2)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)dA \\ &= \iint_R (2x - x)dA \\ &= \int_0^1\int_0^{2x} xdydx = \int_0^2\int_{\frac{y}{2}}^1 xdxdy = \frac{2}{3}\end{aligned}$$

Example:

Find the area of the region bounded by the hypocycloid with vector equation

$$\vec{r}(t) = \cos^3 t\,\vec{i} + \sin^3 t\,\vec{j}, \qquad 0 \le t \le 2\pi.$$

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint xdy - ydx$$

We have $x = \cos^3 t, dx = 3\cos^2 t(-\sin t)dt$, and $y = \sin^3 t, dy = 3\sin^2 t\cos t dt$.

Using $A = \oint_C xdy : A = \oint_C xdy = \int_0^{2\pi} \cos^3 t(3\sin^2 t\cos t)dt = \frac{3\pi}{8}$.

Example:

Evaluate

$$\oint_C (1 + 10xy + y^2)dx + (6xy + 5x^2)dy$$

where C is the square with vertices $(0,0),(a,0),(a,a),(0,a)$ with counterclockwise orientation.

Solution: We use Green's Theorem since the path is closed. Now

$$P(x,y) = 1 + 10xy + y^2 \text{ and } Q(x,y) = 6xy + 5x^2$$

Thus

$$P_y = 10x + 2y \text{ and } Q_x = 6y + 10x$$

$$Q_x - P_y = 4y$$

By Green's Theorem

$$\begin{aligned}\oint_C (1 + 10xy + y^2)dx + (6xy + 5x^2)dy &= \iint_R (Q_x - P_y)dA \\ &= \int_0^a\int_0^a 4ydxdy = 2a^3\end{aligned}$$

Example:

Evaluate

$$\oint_C x^2ydx - xy^2dy$$

where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.
Solution: Since the path is a closed curve, we may use Green's Theorem to evaluate the line integral. Thus

$$\oint_C x^2y\,dx - xy^2\,dy = \iint\limits_{x^2+y^2\le 4} \left(\frac{\partial(-xy^2)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\right)dA$$
$$= \iint\limits_{x^2+y^2\le 4} (-y^2 - x^2)\,dA = -\int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta$$
$$= -8\pi$$

Example:
Verify Green's theorem for

$$\oint_C (4x - 2y)\,dx + (2x + 6y)\,dy$$

where C is the ellipse $x = 2\cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$. (Recall that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.)

SOLUTION

For this ellipse, $a = 2$ and $b = 1$. Let G be the interior of C. Green's theorem states that the two integrals $\oint_C P\,dx + Q\,dy$ and $\iint_G (Q_x - P_y)\,dx\,dy$ are equal. We must verify this.

Since $Q_x = 2$ and $P_y = -2$,

$$\iint_G (Q_x - P_y)\,dx\,dy = \iint_G 4\,dx\,dy = 4\iint_G dx\,dy = 4(Area of G) = 4(\pi)(2)(1) = 8\pi$$

The ellipse is already parametrized by θ. Since $dx = -2\sin\theta\,d\theta$ and $dy = \cos\theta\,d\theta$,

$$\oint_C P\,dx + Q\,dy = \oint_C (4x - 2y)\,dx + (2x + 6y)\,dy$$
$$= \int_0^{2\pi} [(8\cos\theta - 2\sin\theta)(-2\sin\theta) + (4\cos\theta + 6\sin\theta)(\cos\theta)]\,d\theta$$
$$= \int_0^{2\pi} [-16\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 6\sin\theta\cos\theta]\,d\theta$$
$$= \int_0^{2\pi} [4 - 10\sin\theta\cos\theta]\,d\theta = 8\pi$$