Ma 227 **Exam I Solutions** **2/19/03**

Name: ________________________________ **ID**: _________________________

Lecture Section: _______ **Recitation Section**: _____

I pledge my honor that I have abided by the Stevens Honor System. __________________________

SHOW ALL WORK! You may not use a calculator on this exam.

1 [20 **pts**.] Does the system

$$\begin{aligned} x_1 + x_2 &= c_1 \\ x_1 - x_2 &= c_2 \\ -x_1 + 2x_2 &= c_3 \end{aligned}$$

possess a solution for all c_1, c_2, c_3? Explain your conclusion.
Solution:

$$\begin{bmatrix} 1 & 1 & c_1 \\ 1 & -1 & c_2 \\ -1 & 2 & c_3 \end{bmatrix} \to_{\substack{-R_1+R_2 \\ R_1+R_3}} \begin{bmatrix} 1 & 1 & c_1 \\ 0 & -2 & c_2 - c_1 \\ 0 & 3 & c_3 + c_1 \end{bmatrix} \to_{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & c_1 \\ 0 & 1 & \frac{c_1 - c_2}{2} \\ 0 & 3 & c_3 + c_1 \end{bmatrix}$$

$$\to_{-3R_2+R_3} \begin{bmatrix} 1 & 1 & c_1 \\ 0 & 1 & \frac{c_1-c_2}{2} \\ 0 & 0 & -\frac{1}{2}c_1 + \frac{3}{2}c_2 + c_3 \end{bmatrix}$$

Thus for the system to have a solution we must have

$$-\frac{1}{2}c_1 + \frac{3}{2}c_2 + c_3 = 0$$

2 Let

$$A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$$

2a [**20 pts**.] Find all eigenvalues and eigenvectors of the matrix A.

Solution: $\det(A - rI) = \begin{vmatrix} -4-r & 2 \\ -3 & 1-r \end{vmatrix} = -(4+r)(1-r) + 6 = r^2 + 3r + 2 = (r+1)(r+2)$

Therefore the eigenvalues are $r = -1, -2$.
For $r = -1$ we have

$$\begin{aligned} -3x_1 + 2x_2 &= 0 \\ -3x_1 + 2x_2 &= 0 \end{aligned}$$

so the eigenvector corresponding to $r = -1$ is $\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

For $r = -2$ we have

$$-2x_1+2x_2=0$$
$$-3x_1+3x_2=0$$

so the eigenvector corresponding to $r=-2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2b [**20 pts**.] Solve the initial value problem

$$x'(t)=Ax(t), \quad x(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where A is the matrix above.

Solution: The two linearly independent solutions are $\begin{bmatrix} 2e^{-t} \\ 3e^{-t} \end{bmatrix}, \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$. The general solution is

$$x(t)=c_1\begin{bmatrix} 2e^{-t} \\ 3e^{-t} \end{bmatrix}+c_2 \begin{matrix} e^{-2t} \\ e^{-2t} \end{matrix}$$

Therefore

$$x(0)=c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix}+c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We have two equations for c_1 and c_2

$$2c_1+c_2=1$$
$$3c_1+c_2=0$$

so $c_1=-1, c_2=3$.

$$x(t)=-\begin{bmatrix} 2e^{-t} \\ 3e^{-t} \end{bmatrix}+3 \begin{matrix} e^{-2t} \\ e^{-2t} \end{matrix}$$

3 a [**20 pts**.] Let

$$A=\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Find A^{-1} using elementary row operations.

Solution: $\begin{bmatrix} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix} \to_{\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix} \to_{-5R_1+R_2} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{5}{3} & 1 \end{bmatrix}$

$\to_{3R_2} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -5 & 3 \end{bmatrix} \to_{-\frac{1}{3}R_2+R_1} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{bmatrix}$

SNb check: $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, inverse: $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

3 b [**20 pts**.] Let

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Find a 2×2 matrix X such that

$$AX = B$$

and a 2×2 matrix Y such that

$$YA = B$$

Solution: To find X multiply both sides by A^{-1}.

$$\left(A^{-1}A\right)X = IX = X = A^{-1}B$$

$$A^{-1}B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}$$

Multiply on the *right* by A^{-1}.

$$Y\left(AA^{-1}\right) = YI = Y = BA^{-1}$$

Thus

$$Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -8 & 5 \\ -14 & 9 \end{bmatrix}$$