## Ma 227 Exam I Solutions 10/11/06

**Name**: _______________________________

**Lecture Section**: _______ **Recitation Section**:______

I pledge my honor that I have abided by the Stevens Honor System.
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**You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.**

Score on Problem #1 ________

#1b ________

#2a ________

#2b ________

#2c ________

Total Score ________

**1** [20 **pts**.] Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Find $A^{-1}$. Be sure to show all the steps in your calculation and indicate what you are doing in each step.

Solution:

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \rightarrow_{-2R_1+R_2} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -5 & -2 & 1 \end{bmatrix} \rightarrow_{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} \end{bmatrix} \rightarrow_{-4R_2+R_1} \begin{bmatrix} 1 & 0 & -\frac{3}{5} & -\frac{4}{5} \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

SNB Check: $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, inverse: $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

**1b** [20 **pts**.] Find the solution to the system

$$\begin{aligned} x_1 + 4x_2 &= c_1 \\ 2x_1 + 3x_2 &= c_2 \end{aligned}$$

Solution: Rewrite the system as

$$AX = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then

$$X = A^{-1}C = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5}c_2 - \frac{3}{5}c_1 \\ \frac{2}{5}c_1 - \frac{1}{5}c_2 \end{bmatrix}$$

so

$$\begin{aligned} x_1 &= \frac{4}{5}c_2 - \frac{3}{5}c_1 \\ x_2 &= \frac{2}{5}c_1 - \frac{1}{5}c_2 \end{aligned}$$

SNB check:

$$\begin{aligned} x_1 + 4x_2 &= c_1 \\ 2x_1 + 3x_2 &= c_2 \end{aligned}$$

, Solution is: $\left[ x_1 = \frac{4}{5}c_2 - \frac{3}{5}c_1, x_2 = \frac{2}{5}c_1 - \frac{1}{5}c_2 \right]$

**2** Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

**2a** [**20 pts**.] Find all eigenvalues and eigenvectors of the matrix $A$.

Solution:

$$\det(A - rI) = \left|\begin{bmatrix} 1-r & 4 \\ 2 & 3-r \end{bmatrix}\right| = (1-r)(3-r) - 8 = 3 - 4r + r^2 - 8$$
$$= r^2 - 4r - 5 = (r-5)(r+1)$$

Therefore the eigenvalues are $r = 5, r = -1$.

To find the eigenvalues consider

$$(1-r)x_1 + 4x_2 = 0$$
$$2x_1 + (3-r)x_2 = 0$$

For $r = 5$ this becomes

$$-4x_2 + 4x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

so $x_1 = x_2$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

For $r = -1$ we have

$$2x_1 + 4x_2 = 0$$
$$2x_1 + 4x_2 = 0$$

so $x_1 = -2x_2$. Thus $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector.

SNB check: $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 5$

**2b** [**20 pts**.] Find a general homogeneous solution of

$$\frac{dx_1}{dt} = x_1 + 4x_2$$
$$\frac{dx_2}{dt} = 2x_1 + 3x_2$$

Solution: The system can be written as

$$X' = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} X$$

From $2a$ the solution is

$$X = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

SNB check:

$$\frac{dx_1}{dt} = x_1 + 4x_2$$
$$\frac{dx_2}{dt} = 2x_1 + 3x_2$$

, Exact solution is: $\left\{\left[x_2(t) = C_4 e^{5t} - \frac{1}{2} C_3 e^{-t}, x_1(t) = C_3 e^{-t} + C_4 e^{5t}\right]\right\}$

$2c$ $\left[20 \textbf{ pts.}\right]$ Find a particular solution of

$$\frac{dx_1}{dt} = x_1 + 4x_2 - e^{2t}$$
$$\frac{dx_2}{dt} = 2x_1 + 3x_2 + e^{2t}$$

Solution: This nonhomogeneous system can be written as

$$X' = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} X + \begin{bmatrix} -e^{2t} \\ e^{2t} \end{bmatrix}$$

Let

$$X_p = \begin{bmatrix} ae^{2t} \\ be^{2t} \end{bmatrix}$$

Then substituting $X_p$ into the system leads to

$$\begin{bmatrix} 2ae^{t} \\ 2be^{2t} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} ae^{2t} \\ be^{2t} \end{bmatrix} + \begin{bmatrix} -e^{2t} \\ e^{2t} \end{bmatrix}$$

or

$$\begin{bmatrix} 2ae^{2t} \\ 2be^{2t} \end{bmatrix} = \begin{bmatrix} ae^{2t} + 4be^{2t} - e^{2t} \\ 2ae^{2t} + 3be^{2t} + e^{2t} \end{bmatrix}$$

Therefore we have the equations

$$2a = a + 4b - 1$$
$$2b = 2a + 3b + 1$$

or

$$a - 4b = -1$$
$$-2a - b = 1$$

, Solution is: $\left[a = -\frac{5}{9}, b = \frac{1}{9}\right]$. Therefore

$$X_p = \begin{bmatrix} -\frac{5}{9} e^{2t} \\ \frac{1}{9} e^{2t} \end{bmatrix}$$

SNB check:

$$\frac{dx_1}{dt} = x_1 + 4x_2 - e^{2t}$$
$$\frac{dx_2}{dt} = 2x_1 + 3x_2 + e^{2t}$$
, Exact solution is: $\left\{\left[x_1(t) = C_{10}e^{5t} - \frac{5}{9}e^{2t} + C_9e^{-t}, x_2(t) = \frac{1}{9}e^{2t} + C_{10}e^{5t} - \frac{1}{2}C_9e^{-t}\right]\right\}$