Ma 227 **Exam I Solutions 10/10/07**

Name: ______________________________

Lecture Section: _______ **Recitation Section**:______

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#2a ________

#2b ________

#2c ________

Total Score ________

1 [25 **pts**.] The matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix}$$

has the eigenvalues 3 and –2. For this matrix –2 is a repeated eigenvalue with multiplicity two. Find three linearly independent eigenvectors for A.

Solution: Note: This example is the last example in the notes dealing with eigenvalues and eigenvectors.

The system $(A - \lambda I)X = 0$ is

$$\begin{aligned} (1-\lambda)x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - (4+\lambda)x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + (2-\lambda)x_3 &= 0 \end{aligned}$$

Setting $\lambda = 3$ yields

$$\begin{aligned} -2x_2 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 7x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 - x_3 &= 0 \end{aligned}$$

To solve this system we form the augmented matrix for this system and row reduce it.

$$\begin{bmatrix} -2 & -2 & 4 & 0 \\ 3 & -7 & 4 & 0 \\ 3 & -2 & -1 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{2}R_1]{-R_2+R_3} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 3 & -7 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{bmatrix} \xrightarrow[\frac{1}{5}R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{10R_3+R_2} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solutions of the above system are also the solutions of the system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Thus $x_1 = x_2 = x_3$ and an eigenvector corresponding to $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Setting $\lambda = -2$ in the system $(A - \lambda I)X = 0$ yields

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \end{aligned}$$

The augmented matrix for this system is $\begin{bmatrix} 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \end{bmatrix} \to_{-R_1+R_3}^{-R_1+R_2} \begin{bmatrix} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to_{\frac{1}{3}R_1} \begin{bmatrix} 1 & -\frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we have the one equation

$$x_1 - \frac{2}{3}x_2 + \frac{4}{3}x_3 = 0$$

To get two linearly independent vectors we first take $x_3 = 0$ and get $x_1 = \frac{2}{3}x_2$. Letting $x_2 = 1$ yields the eigenvector $\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$. To get a second vector we set $x_2 = 0$ and get $x_1 = -\frac{4}{3}x_3$. Letting $x_3 = 1$ yields the eigenvector $\begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$.

2 For the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

SNB gives: eigenvectors: $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 3$

2a [**20 pts**.] Find a general homogeneous solution of

$$\frac{dx_1}{dt} = x_1 + 2x_2$$
$$\frac{dx_2}{dt} = 2x_1 + x_2$$

Solution: The system can be written as $x'(t) = Ax(t)$, where A is the matrix given above. Since we are given the eigenvalues and eigenvectors, then

$$x_h(t) = c_1 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$2b$ [30 **pts**.] Find a general solution of the nonhomogeneous system

$$\frac{dx_1}{dt} = x_1 + 2x_2 - e^t$$
$$\frac{dx_2}{dt} = 2x_1 + x_2 + 3$$

Solution: We may write the system as

$$x'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -e^t \\ 3 \end{bmatrix}$$

We assume

$$x_p(t) = \begin{bmatrix} a + be^t \\ c + de^t \end{bmatrix}$$

Then

$$x_p'(t) = \begin{bmatrix} be^t \\ de^t \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a + be^t \\ c + de^t \end{bmatrix} + \begin{bmatrix} -e^t \\ 3 \end{bmatrix}$$

which implies

$$\begin{bmatrix} be^t \\ de^t \end{bmatrix} = \begin{bmatrix} a + 2c + be^t + 2de^t \\ 2a + c + 2be^t + de^t \end{bmatrix} + \begin{bmatrix} -e^t \\ 3 \end{bmatrix}$$

Thus by lining up the constant terms we have

$$a + 2c = 0$$
$$2a + c = -3$$

So $[a = -2, c = 1]$.

For the terms containing e^t we have

$$b = b + 2d - 1$$
$$d = 2b + d$$

Therefore, $b = 0$ and $d = \frac{1}{2}$.

$$x_p(t) = \begin{bmatrix} -2 \\ 1 + \frac{1}{2}e^t \end{bmatrix}$$

Finally

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 + \frac{1}{2}e^t \end{bmatrix}$$

SNB check:

$$\frac{dx_1}{dt} = x_1 + 2x_2 - e^t$$
$$\frac{dx_2}{dt} = 2x_1 + x_2 + 3$$

, Exact solution is: $\left\{\left[x_1(t) = C_5 e^{-t} + C_6 e^{3t} - 2, x_2(t) = \frac{1}{2}e^t - C_5 e^{-t} + C_6 e^{3t} + 1\right]\right\}$

3 [25 **pts**.] Find the function matrix $X^{-1}(t)$ whose value at t is the inverse of the matrix

$$X(t) = \begin{bmatrix} e^{3t} & 1 & t \\ 3e^{3t} & 0 & 1 \\ 9e^{3t} & 0 & 0 \end{bmatrix}$$

Solution: We form

$$\begin{bmatrix} e^{3t} & 1 & t & 1 & 0 & 0 \\ 3e^{3t} & 0 & 1 & 0 & 1 & 0 \\ 9e^{3t} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to_{\substack{-3R_1+R_2 \\ -9R_1+R_3}} \begin{bmatrix} e^{3t} & 1 & t & 1 & 0 & 0 \\ 0 & -3 & 1-3t & -3 & 1 & 0 \\ 0 & -9 & -9t & -9 & 0 & 1 \end{bmatrix} \to_{-3R_2+R_3} \begin{bmatrix} e^{3t} & 1 & t & 1 & 0 & 0 \\ 0 & -3 & 1-3t & -3 & 1 & 0 \\ 0 & 0 & -3 & 0 & -3 & 1 \end{bmatrix}$$

$$\to_{\substack{-\frac{1}{3}R_2 \\ -\frac{1}{3}R_3}} \begin{bmatrix} e^{3t} & 1 & t & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3}+t & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 & -\frac{1}{3} \end{bmatrix} \to_{-R_2+R_1} \begin{bmatrix} e^{3t} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3}+t & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 & -\frac{1}{3} \end{bmatrix} \to_{\substack{-\frac{1}{3}R_3+R_1 \\ \left(\frac{1}{3}-t\right)R_3+R_2}}$$

$$\to_{e^{-3t}R_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{9}e^{-3t} \\ 0 & 1 & 0 & 1 & -t & -\frac{1}{9}+\frac{1}{3}t \\ 0 & 0 & 1 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

Therefore

$$X^{-1}(t) = \begin{bmatrix} 0 & 0 & \frac{1}{9}e^{-3t} \\ 1 & -t & -\frac{1}{9}+\frac{1}{3}t \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}$$