Ma 227 **Exam I Solutions 10/14/09**

Name: ________________________________

Lecture Section: _______ **Recitation Section**:______

I pledge my honor that I have abided by the Stevens Honor System.
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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2a ________

#2b ________

#3 ________

Total Score ________

1 [25 **pts**.] Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Show that the eigenvectors you find are linearly independent.
Solution:

$$\begin{vmatrix} 1-r & 1 \\ -2 & 4-r \end{vmatrix} = (1-r)(4-r)+2 = r^2-5r+6 = (r-2)(r-3)$$

Thus the eigenvalues are $r = 2, 3$. The system $(A - rI)X = 0$ is

$$(1-r)x_1 + x_2 = 0$$
$$-2x_1 + (4-r)x_2 = 0$$

For $r = 2$ we have

$$-x_1 + x_2 = 0$$
$$-2x_1 + 2x_2 = 0$$

Thus $x_1 = x_2$ and a corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $r = 3$ we have

$$-2x_1 + x_2 = 0$$
$$-2x_1 + x_2 = 0$$

so $x_2 = 2x_1$ and a corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$$

These vectors are L.I.

SNB check $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right\} \leftrightarrow 3$

2 The eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

are $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 3$

2a [25 **pts**.] Solve the initial value problem

$$AX = 0, \ \ X(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Solution:

$$X_h(t) = c_1 \begin{bmatrix} e^t \\ -e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

$$X(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Thus

$$c_1 + c_2 = 2$$
$$-c_1 + c_2 = 0$$

and $c_1 = c_2 = 1.$ The solution is

$$X(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \end{bmatrix} + \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} = \begin{bmatrix} e^t + e^{3t} \\ -e^t + e^{3t} \end{bmatrix}$$

2b [30 **pts**.] Find a general solution of the nonhomogeneous system

$$X' = AX + \begin{bmatrix} 15e^{-2t} \\ 0 \end{bmatrix}$$

Solution: Let

$$X_p(t) = \begin{bmatrix} a_1 e^{-2t} \\ a_2 e^{-2t} \end{bmatrix}$$

Then the DE implies

$$\begin{bmatrix} -2a_1 e^{-2t} \\ -2a_2 e^{-2t} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 e^{-2t} \\ a_2 e^{-2t} \end{bmatrix} + \begin{bmatrix} 15e^{-2t} \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -2a_1e^{-2t} \\ -2a_2e^{-2t} \end{bmatrix} = \begin{bmatrix} 2a_1e^{-2t} + a_2e^{-2t} + 15e^{-2t} \\ a_1e^{-2t} + 2a_2e^{-2t} \end{bmatrix}$$

Thus

$$-4a_1 - a_2 = 15$$
$$a_1 + 4a_2 = 0$$

So $a_1 = -4a_2$ and $16a_2 - a_2 = 15$ so $a_2 = 1$ and $a_1 = -4.$ Thus

$$X_p(t) = \begin{bmatrix} -4e^{-2t} \\ e^{-2t} \end{bmatrix}$$

and

$$X(t) = X_h(t) + X_p(t) = c_1\begin{bmatrix} e^t \\ -e^t \end{bmatrix} + c_2\begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} -4e^{-2t} \\ e^{-2t} \end{bmatrix}$$

3 20 **pts**. Find, if possible, the solution(s) to the system

$$x + 2y + 6u + 11v = 0$$
$$x + y + 3u + 6v = 1$$
$$x + 3y + 9u + 16v = 6$$
$$x + 4y + 12u + 21v = 8$$

Justify your results.

Solution: $$\begin{bmatrix} 1 & 2 & 6 & 11 & 0 \\ 1 & 1 & 3 & 6 & 1 \\ 1 & 3 & 9 & 16 & 6 \\ 1 & 4 & 12 & 21 & 8 \end{bmatrix} \underset{\substack{-R_1+R_3 \\ -R_1+R_4}}{\to^{-R_1+R_2}} \begin{bmatrix} 1 & 2 & 6 & 11 & 0 \\ 0 & -1 & -3 & -5 & 1 \\ 0 & 1 & 3 & 5 & 6 \\ 0 & 2 & 6 & 10 & 8 \end{bmatrix} \to^{R_2+R_3} \begin{bmatrix} 1 & 2 & 6 & 11 & 0 \\ 0 & -1 & -3 & -5 & 1 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 2 & 6 & 10 & 8 \end{bmatrix}$$

The third row implies $0x_1 + 0x_2 + 0x_3 + 0x_4 = 20,$ which is impossible. Hence there is no solution.