Ma 227 **Exam I Solutions** **10/13/10**

Name: ________________________________

Lecture Section: _______ **Recitation Section**:______

I pledge my honor that I have abided by the Stevens Honor System.

____

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2a ________

#2b ________

#3 ________

Total Score ________

1 $\left[25\ \textbf{pts}.\right]$ Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix}$$

Show that the eigenvectors you find are linearly independent.
Solution:

$$\begin{vmatrix} 6-r & 3 \\ -2 & -1-r \end{vmatrix} = -(6-r)(1+r)+6 = -\left(6+5r-r^2\right)+6 = r^2-5r$$

Thus the eigenvalues are $r = 0, 5$.

The system of equations $(A - rI)X = 0$ is

$$\begin{aligned} (6-r)x_1 + 3x_2 &= 0 \\ -2x_1 - (1+r)x_2 &= 0 \end{aligned}$$

For $r = 0$ the equations are

$$\begin{aligned} 6x_2 + 3x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$$

so $2x_1 = -x_2$ and an eigenvector for $r = 0$ is $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$. For $r = 5$ we have

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ -2x_1 - 6x_2 &= 0 \end{aligned}$$

so $x_1 = -3x_2$. Thus an eigenvector for $r = 5$ is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

SNB check $\begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right\} \leftrightarrow 0, \left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\} \leftrightarrow 5$

Since

$$\begin{vmatrix} -\frac{1}{2} & -3 \\ 1 & 1 \end{vmatrix} = 2.5 \neq 0$$

the vectors are linearly independent.

2 The eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

are $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow 2, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \leftrightarrow 3$

$2a$ [25 **pts**.] Solve the initial value problem

$$x' = Ax, \quad x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

Solution:

$$x_h(t) = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{2}e^{3t} \\ e^{3t} \end{bmatrix}$$

$$x_h(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Thus

$$c_1 + \frac{1}{2}c_2 = 2$$
$$c_1 + c_2 = 0$$

, Solution is: $[c_1 = 4, c_2 = -4]$
and

$$x(t) = 4 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} - 4 \begin{bmatrix} \frac{1}{2}e^{3t} \\ e^{3t} \end{bmatrix}$$

$2b$ [30 **pts**.] Find a general solution of the nonhomogeneous system

$$x' = Ax + \begin{bmatrix} 0 \\ 4e^{-t} \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

Solution: Let

$$x_p(t) = \begin{bmatrix} a_1 e^{-t} \\ a_2 e^{-t} \end{bmatrix}$$

Then the DE implies

$$\begin{bmatrix} -a_1 e^{-t} \\ -a_2 e^{-t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} a_1 e^{-t} \\ a_2 e^{-t} \end{bmatrix} + \begin{bmatrix} 0 \\ 4e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} -a_1 e^{-t} \\ -a_2 e^{-t} \end{bmatrix} = \begin{bmatrix} a_1 e^{-t} + a_2 e^{-t} \\ 4e^{-t} - 2a_1 e^{-t} + 4a_2 e^{-t} \end{bmatrix}$$

Thus

$$-a_1 = a_1 + a_2$$
$$-a_2 = -2a_1 + 4a_2 + 4$$

$$2a_1 + a_2 = 0$$
$$2a_1 - 5a_2 = 4$$

, Solution is: $\left[a_1 = \frac{1}{3}, a_2 = -\frac{2}{3}\right]$. Thus

$$x_p(t) = \begin{bmatrix} \frac{1}{3}e^{-t} \\ -\frac{2}{3}e^{-t} \end{bmatrix}$$

The general solution is

$$x(t) = x_h(t) + x_p(t) = c_1\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + c_2\begin{bmatrix} \frac{1}{2}e^{3t} \\ e^{3t} \end{bmatrix} + \begin{bmatrix} \frac{1}{3}e^{-t} \\ -\frac{2}{3}e^{-t} \end{bmatrix}$$

3 [20 **pts**.] Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution: We form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and row reduce the left hand 3×3. Multiplying row one by -1 and adding to row two yields

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Multiply row two by -1 and add it to row three

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Add -1 times row two to row one

$$\begin{bmatrix} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Mutliply row three by –1

$$\begin{bmatrix} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Add row three to row one and add –2 times row three to row one

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Therefore the inverse is $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$.