Ma 227 **Exam I A Solutions** **10/3/13**

Name: ________________________________

Lecture Section: ______

I pledge my honor that I have abided by the Stevens Honor System. ____

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

There is a table of integrals on the last page of the exam.

Score on Problem #1 ________

#2a ________

#2b ________

#3 ________

#4 ________

Total Score ________

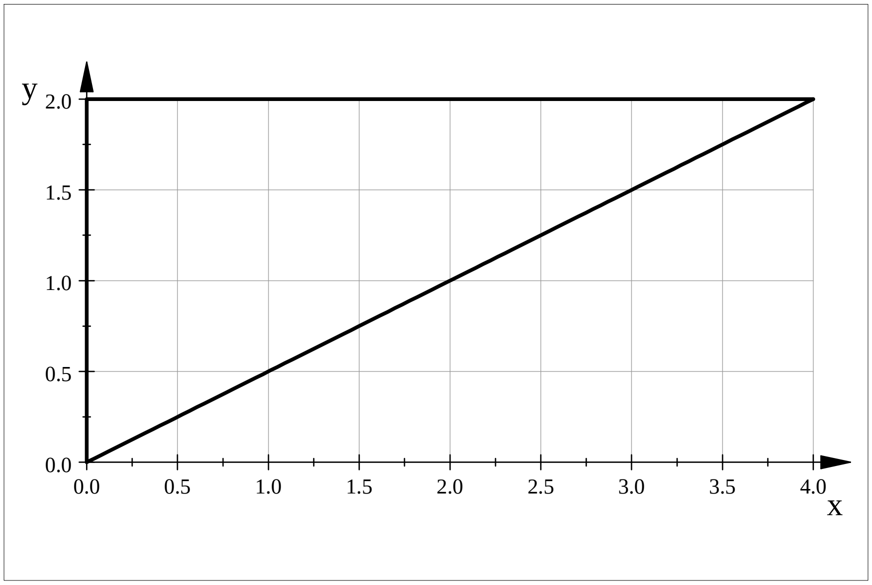
1 [25 **pts**.] Evaluate the integral

$$\iint_D e^{y^2} dA$$

where D is the triangle bounded by the lines $y = \frac{x}{2}, y = 2,$ and $x = 0.$ Sketch D.

Solution: The lines $y = \frac{x}{2}, y = 2$ intersect at $(4, 2)$. Thus D is shown below.

$\frac{x}{2}$



We cannot do the y integration first. Thus

$$\iint_D e^{y^2} dA = \int_0^2 \int_0^{2y} e^{y^2} dxdy$$
$$= \int_0^2 2ye^{y^2} dy = e^{y^2} \Big|_0^2 = e^4 - 1$$

2 a $\left[\right.$20 **pts**.$\left.\right]$ Calculate the

$$\iint_R \sin\sqrt{x^2 + y^2}\, dA$$

The region R is the disk $x^2 + y^2 \le \pi^2$.

Solution: We switch to polar coordinates. Then we have (Using an integral from the table.)

$$\iint_R \sin\sqrt{x^2+y^2}\, dA = \iint_R \sin\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\, r dr d\theta$$

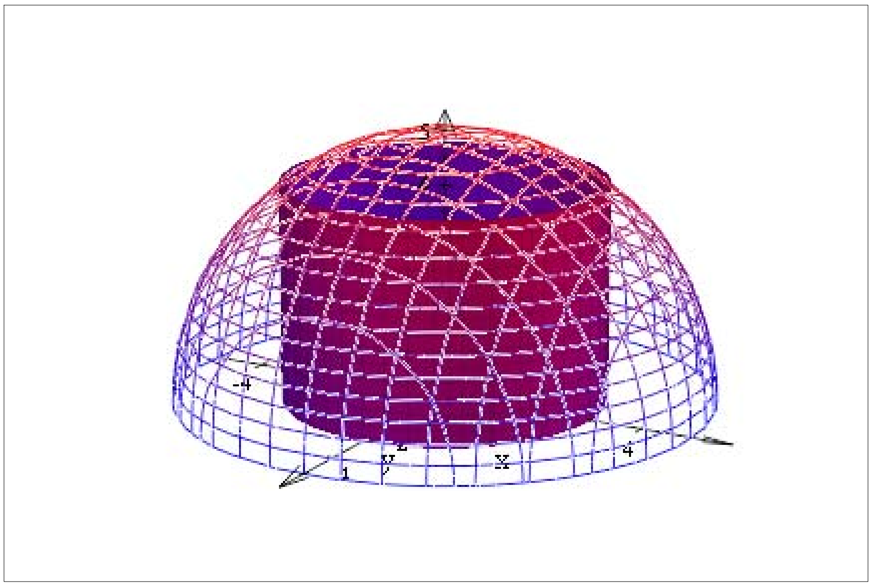
$$= \int_0^{2\pi}\int_0^{\pi} \sin r(r) dr d\theta = \int_0^{2\pi} [\sin r - r\cos r]_0^{\pi} d\theta$$

$$= 2\pi[\pi] = 2\pi^2$$

2 b [15 **pts.**] Give an integral in rectangular coordinates for the surface area of the portion of the hemisphere $x^2+y^2+z^2 = 25,\ z \geq 0$ that lies above the circular region $x^2+y^2 \leq 9$ in the x,y –plane. Sketch the surface area to be found. DO NOT EVALUATE THIS INTEGRAL.

Solution:

$x^2+y^2+z^2 = 25$



$$\text{Surface Area} = \iint_R \sqrt{1+(z_x)^2+(z_y)^2}\, dA$$

where R is the projection of the surface onto the x,y –plane. The equation of the hemisphere can be written as $z^2 = 25 - x^2 - y^2$. Differentiating we have $2zz_x = -2x$ or $z_x = -\frac{x}{z}$. Similarly, $z_y = -\frac{y}{z}$. Therefore

$$\sqrt{1+(z_x)^2+(z_y)^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \frac{1}{z}\sqrt{z^2+x^2+y^2} = \frac{5}{\sqrt{25-x^2-y^2}}$$

$$\text{Surface Area} = \iint_{x^2+y^2\leq 9} \frac{5}{\sqrt{25-x^2-y^2}} dA$$

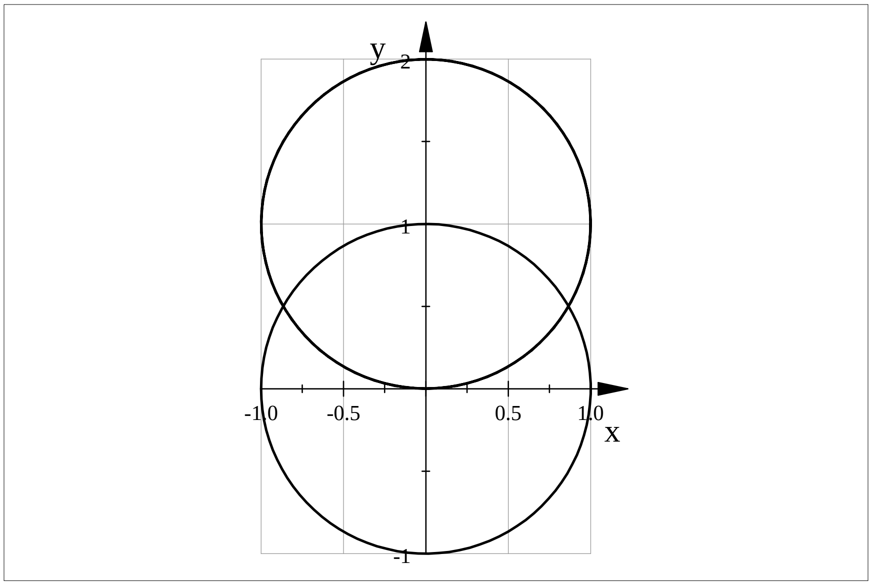
$$= \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{5}{\sqrt{25-x^2-y^2}} dx dy = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25-x^2-y^2}} dy dx$$

3 [25 **pts.**] Give a triple integral in cylindrical coordinates for the volume of the of the solid region cut from the sphere

$$x^2 + y^2 + z^2 = 1$$

by the cylinder $r = 2\sin\theta$. Sketch the solid. DO NOT EVALUATE THIS INTEGRAL.

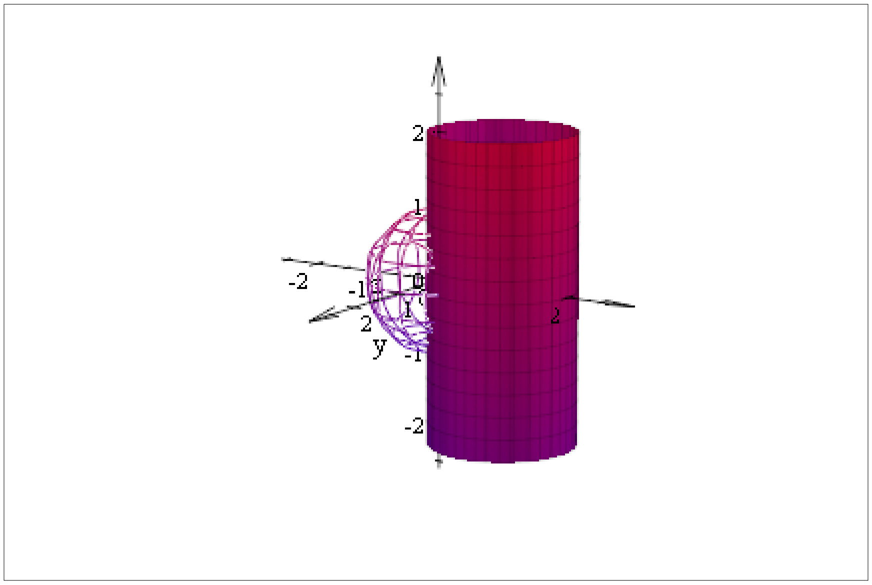
Solution: In the x, y –plane $r = 2\sin\theta$ is the circle $x^2 + (y-1)^2 = 1$ shown below. Also the sphere intersects the x, y –plane in the circle $r = 1$. These circles are shown in the diagram below.
$2\sin\theta$



The circles intersect at

$$2\sin\theta = 1$$

or $\sin\theta = \frac{1}{2}$, that is $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Therefore from the above picture there are three regions of integration needed.
The volume is shown below.
$x^2 + y^2 + z^2 = 1$



We shall use cylindrical coordinates. In cylindrical coordinates the equation of the sphere is $r^2 + z^2 = 1$.

$$-\sqrt{1-r^2} \leq z \leq \sqrt{1-r^2}$$

The volume is given by

$$V = \int_0^{\frac{\pi}{6}} \int_0^{2\sin\theta} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr d\theta + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr d\theta + \int_{\frac{5\pi}{6}}^{\pi} \int_0^{2\sin\theta} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr d\theta$$

4 [20 **pts**.] Give an integral in **spherical** coordinates for the volume of the solid that lies above the cone

$\phi = \frac{\pi}{6}$ and below the sphere $\rho = 2a\cos\phi$. Sketch the volume. DO NOT EVALUATE THIS INTEGRAL.

Solution: The equation of the sphere may be written as $\rho^2 = 2a\rho\cos\phi$ which in Cartesian coordinates is

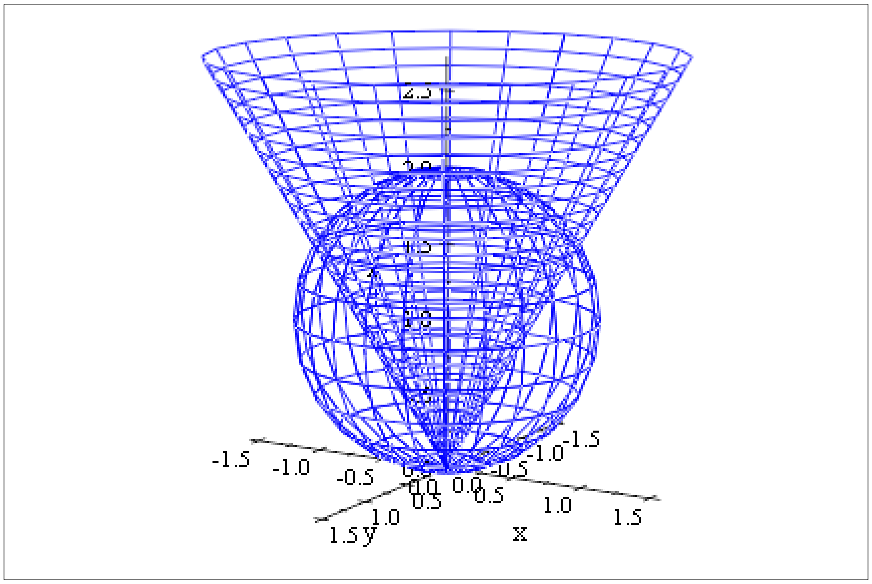
$$x^2 + y^2 + z^2 = 2az$$

or

$$x^2 + y^2 + (z-a)^2 = a^2$$

Thus the sphere has center at $(0, 0, a)$ and radius a.

$\left(\rho, \theta, \frac{\pi}{6}\right)$



The volume is given by

$$\text{Volume} = \iiint\limits_V dV = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2a\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Table of Integrals

$\int x \sin x dx = \sin x - x \cos x + C$
$\int x \cos x dx = \cos x + x \sin x + C$
$\int \sin^2 x dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} x + C$
$\int \cos^2 x dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} x + C$
$\int \sin^3 x dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C$
$\int \cos^3 x dx = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C$
$\int \sin^4 x dx = \frac{3}{8} x - \frac{3}{16} \pi - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$
$\int \cos^4 x dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$
$\int t e^{at} dt = \frac{1}{a^2} e^{at} (at - 1) + C$
$\int t^2 e^{at} dt = \frac{1}{a^3} e^{at} \left(a^2 t^2 - 2at + 2 \right) + C$