Ma 227 **Exam IB Solutions** **3/3/04**

Name: ______________________________ **ID**: ________________________

Lecture Section: _______ **Recitation Section**:______

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

Total Score ________

1 [20 **pts**.] Let

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

Find A^{-1}. Be sure to show all the steps in your calculation and indicate what you are doing in each step.

Solution:

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix} \to_{\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix} \to_{-4R_1+R_2} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{4}{3} & 1 \end{bmatrix} \to_{-(2)R_2+R_1} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & \frac{1}{3} & -\frac{4}{3} & 1 \end{bmatrix}$$

$$\to_{3R_2} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{bmatrix} \quad \text{Thus } A^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

Check: Using SNB we have $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$, inverse: $\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$.

2 Let

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

2a [**20 pts**.] Find all eigenvalues and eigenvectors of the matrix A.

Solution: $\det\begin{bmatrix} 2-r & -4 \\ -1 & -1-r \end{bmatrix} = (2-r)(-1-r)-4 = r^2 - r - 6 = (r-3)(r+2) = 0$ and the eigenvalues are $r = 3, -2$. The system of equations $(A - rI)U = 0$ is

$$(2-r)u_1 - 4u_2 = 0$$
$$-u_1 + (-1-r)u_2 = 0$$

For $r = 3$ this is

$$-1u_1 - 4u_2 = 0$$
$$-u_1 - 4u_2 = 0$$

Thus $u_1 = -4u_2$ and an eigenvector corresponding to $r = 3$ is $U_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix}$.

For $r = -2$ we have

$$4u_1 - 4u_2 = 0$$
$$-u_1 + u_2 = 0$$

so $u_1 = u_2$ and an eigenvector corresponding to $r = -2$ is $U_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2b [20 **pts**.] Find a general homogeneous solution of

$$x'(t) = Ax(t)$$

where A is the matrix above in part 2a.

Solution: Two linearly independent homogeneous solutions are $e^{3t}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so

$$x_h(t) = c_1 e^{3t}\begin{bmatrix} -4 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4e^{3t} & e^{-2t} \\ e3^t & e^{-2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$2c$ [20 **pts**.] Find a general solution of

$$x'(t) = Ax(t) + \begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix}$$

where A is the matrix above in part $2a$.

Solution: Let $x_p(t) = e^{-3t}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Then

$$e^{-3t}\begin{bmatrix} -3a_1 \\ -3a_2 \end{bmatrix} = e^{-3t}\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or

$$\begin{bmatrix} -3a_1 \\ -3a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 - 4a_2 + 1 \\ -a_1 - a_2 - 1 \end{bmatrix}$$

or

$$\begin{aligned} 5a_1 - 4a_2 &= -1 \\ -a_1 + 2a_2 &= 1 \end{aligned}$$

, Solution is: $\left[a_1 = \frac{1}{3}, a_2 = \frac{2}{3}\right]$ and

$$x_p(t) = e^{-3t}\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

so

$$x(t) = x_h(t) + x_p(t) = c_1 e^{3t}\begin{bmatrix} -4 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t}\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -4e^{3t} & e^{-2t} \\ e3^t & e^{-2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}e^{-3t} \\ \frac{2}{3}e^{-3t} \end{bmatrix}$$

SNB gives: $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 + e^{-3t} \\ -x_1 - x_2 - e^{-3t} \end{bmatrix}$

$$\begin{aligned} x_1' &= 2x_1 - 4x_2 + e^{-3t} \\ x_2' &= -x_1 - x_2 - e^{-3t} \end{aligned}$$

, Exact solution is: $\left[x_1(t) = \frac{1}{3}e^{-3t} + C_5 e^{-2t} - 4C_6 e^{3t}, x_2(t) = \frac{2}{3}e^{-3t} + C_5 e^{-2t} + C_6 e^{3t}\right]$

3 [**20 pts**.] Rewrite the scalar equation

$$y'' - ty' + 10y = \cos t$$

as a first-order system in normal form.

Solution: Let $x_1(t) = y(t), x_2(t) = y'(t)$. With these substitutions and noting that $y'' = ty' - 10y + \cos t$ the DE becomes the system

$$x_1'(t) = 0x_1(t) + x_2(t)$$
$$x_2'(t) = -10x_1(t) + tx_2(t) + \cos t$$

we have

$$x'(t) = Ax(t) + f(t)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ -10 & t \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$

or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$