Ma 227 **Exam IB Solutions** **2/28/05**

Name: ______________________________ **ID**: ________________________

Lecture Section: _______ **Recitation Section**:______

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

Total Score ________

1 [20 **pts**.] Let

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$$

Find A^{-1}. Be sure to show all the steps in your calculation and indicate what you are doing in each step.

Solution: $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 5 & 7 & 0 & 1 \end{bmatrix} \rightarrow_{-5R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -5 & 1 \end{bmatrix}$

$\rightarrow_{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{5}{3} & -\frac{1}{3} \end{bmatrix} \rightarrow_{-2R_2+R_1} \begin{bmatrix} 1 & 0 & -\frac{7}{3} & \frac{2}{3} \\ 0 & 1 & \frac{5}{3} & -\frac{1}{3} \end{bmatrix}$

Thus

$$A^{-1} = \begin{matrix} -\frac{7}{3} & \frac{2}{3} \\ \frac{5}{3} & -\frac{1}{3} \end{matrix}$$

SNB check: $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$, inverse: $\begin{bmatrix} -\frac{7}{3} & \frac{2}{3} \\ \frac{5}{3} & -\frac{1}{3} \end{bmatrix}$.

2 Let

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

2a [**20 pts**.] Find all eigenvalues and eigenvectors of the matrix A.

Solution: $\begin{vmatrix} 3-r & 1 \\ 1 & 3-r \end{vmatrix} = (3-r)^2 - 1 = r^2 - 6r + 8 = (r-4)(r-2) = 0$ so the eigenvalues are $r = 2, 4$.

Thus we have the system of equations

$$\begin{bmatrix} 2-r & 1 \\ 1 & 2-r \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$(2-r)x_1 + x_2 = 0$$
$$x_1 + (2-r)x_2 = 0$$

For $r = 2$ the system becomes

$$x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

so $x_2 = -x_1$ and we have the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. For $r = 4$ the system becomes

$$-x_1 + x_2 = 0$$
$$x_1 - x_2 = 0$$

so $x_1 = x_2$ and we have the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2b [**20 pts**.] Find a general homogeneous solution of

$$\frac{dx_1}{dt} = 3x_1 + x_2$$
$$\frac{dx_2}{dt} = x_1 + 3x_2$$

Solution: The system can be rewritten as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From 2a we have that the two linearly independent homogeneous solutions are $e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus

$$x_h(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e2^t & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$2c$ $\left[20\ \textbf{pts}.\right]$ Find a general solution of

$$\frac{dx_1}{dt} = 3x_1 + x_2 + 8t - 1$$
$$\frac{dx_2}{dt} = x_1 + 3x_2 - 1$$

Solution: The nonhomogeneous system can be rewritten as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 8t-1 \\ -1 \end{bmatrix}$$

Let $x_p(t) = t\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix}$. Then substituting into the system leads to

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} + \begin{bmatrix} 8t-1 \\ -1 \end{bmatrix}\right)$$

or

$$a_1 = (3a_1 + a_2 + 8)t + (3b_1 + b_2 - 1)$$
$$a_2 = (a_1 + 3a_2)t + (b_1 + 3b_2 - 1)$$

Equating the coefficients of t and the constant terms yields the system

$$
\begin{aligned}
3a_1 + a_2 &= -8\\
a_1 + 3a_2 &= 0\\
a_1 - 3b_1 - b_2 &= -1\\
a_2 - b_1 - 3b_2 &= -1
\end{aligned}
$$

, Solution is: $[a_1 = -3, a_2 = 1, b_1 = -1, b_2 = 1]$ and

$$x_p(t) = \begin{bmatrix} -3t-1 \\ t+1 \end{bmatrix}$$

so

$$x(t) = x_h(t) + x_p(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3t-1 \\ t+1 \end{bmatrix}$$

SNB check:

$$
\begin{aligned}
\frac{dx_1}{dt} &= 3x_1 + x_2 + 8t - 1\\
\frac{dx_2}{dt} &= x_1 + 3x_2 - 1
\end{aligned}
$$

, Exact solution is: $\left[x_1(t) = C_2 e^{4t} - C_1 e^{2t} - 3t - 1, x_2(t) = t + C_1 e^{2t} + C_2 e^{4t} + 1\right]$.

3 [20 **pts**.] Find all solutions, if they exist, of

$$
\begin{aligned}
x_1 - 3x_2 + 4x_3 + 4x_4 &= 1\\
2x_1 - 5x_2 + 6x_3 + 6x_4 &= 1\\
3x_1 - 7x_2 + 9x_3 + 8x_4 &= 3
\end{aligned}
$$

Solution: The augmented matrix for this system is

$$\begin{bmatrix} 1 & -3 & 4 & 4 & 1 \\ 2 & -5 & 6 & 6 & 1 \\ 3 & -7 & 9 & 8 & 3 \end{bmatrix}$$

We perform row operations to obtain the solution.

$$\begin{bmatrix} 1 & -3 & 4 & 4 & 1 \\ 2 & -5 & 6 & 6 & 1 \\ 3 & -7 & 9 & 8 & 3 \end{bmatrix} \xrightarrow{\substack{-2R_1+R_2 \\ -3R_1+R3}} \begin{bmatrix} 1 & -3 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 & -1 \\ 0 & 2 & -3 & -4 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & -3 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 & -1 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\substack{R_1-4R_3 \\ R_2+2R3}} \begin{bmatrix} 1 & -3 & 0 & 4 & -4 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{3R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Using SNB to check we get , row echelon form: $\begin{bmatrix} 1 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$

Thus the solutions are given by

$$x_1 = 2x_4 + 2$$
$$x_3 = 2x_4 + 3$$
$$x_3 = 2$$

Where x_2 is arbitrary.

Using SNB we get

$$x_1 - 3x_2 + 4x_3 + 4x_4 = 1$$
$$2x_1 - 5x_2 + 6x_3 + 6x_4 = 1$$
$$3x_1 - 7x_2 + 9x_3 + 8x_4 = 3$$

, Solution is: $[x_1 = 2x_4 + 2, x_2 = 2x_4 + 3, x_3 = 2]$