Ma 227 **Exam IB Solutions** **2/27/06**

Name: ________________________________

Lecture Section: _______ **Recitation Section**:______

I pledge my honor that I have abided by the Stevens Honor System.

____

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

Total Score ________

1 [20 **pts**.] Let

$$A = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}$$

Find A^{-1}. Be sure to show all the steps in your calculation and indicate what you are doing in each step.

Solution:

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix} \to_{4R_1+R_2} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 6 & 4 & 1 \end{bmatrix} \to_{\frac{1}{6}R_2} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{1}{6} \end{bmatrix}$$

$$\to_{R_2-R_1} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{1}{6} \\ 0 & 1 & \frac{2}{3} & \frac{1}{6} \end{bmatrix}$$

Therefore $A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{6} \end{bmatrix}$

2 Let

$$A = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}$$

2a [**20 pts**.] Find all eigenvalues and eigenvectors of the matrix A.

Solution:

$$\begin{vmatrix} -1-r & 1 \\ 4 & 2-r \end{vmatrix} = (-1-r)(2-r) - 4 = r^2 - r - 6 = 0$$

so we have to solve $r^2 - r - 6 = (r-3)(r+2) = 0$. Thus the eigenvalues are $r = -2, 3$.
The system of equations for the eigenvectors is

$$(-1-r)x_1 + x_2 = 0$$
$$4x_1 + (2-r)x_2 = 0$$

Setting $r = -2$ gives

$$x_1 + x_2 = 0$$
$$4x_1 + 4x_2 = 0$$

Therefore $x_1 = -x_2$ and an eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For $r = 3$ we have

$$-4x_1 + x_2 = 0$$
$$4x_1 - x_2 = 0$$

so $x_1 = \frac{1}{4}x_2$. Thus an eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

2b [**20 pts**.] Find a general homogeneous solution of $\frac{dx_1}{dt} = -x_1 + x_2 \qquad x_1(0) = -2$

$\frac{dx_2}{dt} = 4x_1 + 2x_2 \qquad x_2(0) = 1$

Solution: This system can be written as $x' = Ax$, where A is the matrix above. Thus

$$x_h = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{3t} = \begin{bmatrix} -c_1 e^{-2t} + c_2 e^{3t} \\ c_1 e^{-2t} + 4c_2 e^{3t} \end{bmatrix}$$

$$x_h(0) = \begin{bmatrix} -c_1 + c_2 \\ c_1 + 4c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$-c_1 + c_2 = -2$$
$$c_1 + 4c_2 = 1$$

Therefore, $c_2 = -\frac{1}{5}, c_1 = \frac{9}{5}$. Thus

$$x_h = -\frac{9}{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} - \frac{1}{5} \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{3t}$$

$2c$ [**20 pts**.] Find a general solution of

$$\frac{dx_1}{dt} = -x_1 + x_2 + e^t$$
$$\frac{dx_2}{dt} = 4x_1 + 2x_2 - 2e^t$$

Solution: Let

$$x_p = \begin{bmatrix} ae^t \\ be^t \end{bmatrix}$$

Then

$$x_p' = x_p$$

The nonhomogeneous system in matrix form is

$$x' = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t$$

Substituting x_p into this we get

$$\begin{bmatrix} a \\ b \end{bmatrix} e^t = \begin{bmatrix} -a + b \\ 4a + 2b \end{bmatrix} e^t + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t$$

This gives the equations

$$a = -a + b + 1$$
$$b = 4a + 2b - 2$$

or

$$2a - b = 1$$
$$4a + b = 2$$

Adding the equations gives $a = \frac{1}{2}$, so $b = 0$. Thus

$$x_p = \begin{bmatrix} \frac{1}{2}e^t \\ 0 \end{bmatrix}$$

and

$$x = x_h + x_p = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{3t} + \begin{bmatrix} \frac{1}{2}e^t \\ 0 \end{bmatrix}$$

Check:

$$\frac{dx_1}{dt} = -x_1 + x_2 + e^t$$
$$\frac{dx_2}{dt} = 4x_1 + 2x_2 - 2e^t$$

, Exact solution is: $\left\{\left[x_1(t) = -C_{11}e^{-2t} + C_{12}e^{3t} + \frac{1}{2}e^t, x_2(t) = C_{11}e^{-2t} + 4C_{12}e^{3t}\right]\right\}$

3 [20 **pts**.] Under what condition or conditions on c_1, c_2, c_3 will the system below have a solution? You need only give the condition. You need not solve the system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= c_1 \\ 2x_1 + 4x_2 + 4x_3 + 3x_4 &= c_2 \\ 4x_1 + 8x_2 + 10x_3 + 5x_4 &= c_3 \end{aligned}$$

Solution: $\begin{bmatrix} 1 & 2 & 3 & 1 & c_1 \\ 2 & 4 & 4 & 3 & c_2 \\ 4 & 8 & 10 & 5 & c_3 \end{bmatrix} \to^{-2R_1+R_2}_{-4R_1+R_3} \begin{bmatrix} 1 & 2 & 3 & 1 & c_1 \\ 0 & 0 & -2 & 1 & c_2 - 2c_1 \\ 0 & 0 & -2 & 1 & c_3 - 4c_1 \end{bmatrix} \to^{-R_2+R_3} \begin{bmatrix} 1 & 2 & 3 & 1 & c_1 \\ 0 & 0 & -2 & 1 & c_2 - 2c_1 \\ 0 & 0 & 0 & 0 & c_3 - 4c_1 - c_2 + 2 \end{bmatrix}$

Thus there will be a solution only if

$$c_3 - 2c_1 - c_2 = 0$$