Ma 227 **Exam IB Solutions** **10/13/11**

Name: ________________________________

Lecture Section: _______ **Recitation Section**:______

I pledge my honor that I have abided by the Stevens Honor System.

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You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2a ________

#2b ________

#3 ________

Total Score ________

1 $\left[25 \textbf{ pts}.\right]$ Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

Show that the eigenvectors you find are linearly independent.
Solution:

$$\begin{vmatrix} 1-r & 2 \\ 3 & 2-r \end{vmatrix} = (1-r)(2-r) - 6 = 2 - 3r + r^2 - 6 = r^2 - 3r - 4 = (r-4)(r+1)$$

Thus the eigenvalues are $r = -1, 4$.
The system of equations $(A - rI)X = 0$ is

$$\begin{aligned} (1-r)x_1 + 2x_2 &= 0 \\ 3x_1 + (2-r)x_2 &= 0 \end{aligned}$$

For $r = -1$ the system becomes

$$\begin{aligned} 2x_1 + 2x_2 &= 0 \\ 3x_1 + 3x_2 &= 0 \end{aligned}$$

Thus $x_1 = -x_2$ and an eigenvector for $r = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $r = 4$ the system becomes

$$\begin{aligned} -3x_1 + 2x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}$$

so $2x_2 = 3x_1$. Thus an eigenvector for $r = 4$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Since

$$\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5 \neq 0$$

these vectors are linearly independent.

SNB check: $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right\} \leftrightarrow 4.$

2 The eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix}$$

are $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow 2, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \leftrightarrow 4.$

$2a$ [25 **pts**.] Solve the initial value problem

$$x' = Ax, \quad x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix}$.

The general solution is

$$x_h(t) = c_1 \begin{bmatrix} 1e^{2t} \\ 1e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{3}e^{4t} \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} & \frac{1}{3}e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$x_h(0) = \begin{bmatrix} 1 & \frac{1}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 1 & \frac{1}{3} & 2 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & 2 \\ 0 & \frac{2}{3} & -2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & 2 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \end{bmatrix}$$

Hence $c_1 = 3$ and $c_2 = 3$

$$x_h(t) = 3 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{3}e^{4t} \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} & \frac{1}{3}e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$2b$ [30 **pts**.] Find a general solution of the nonhomogeneous system

$$x' = Ax + \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix}$.

Solution: Let

$$x_p(t) = \begin{bmatrix} a_1 e^{-t} \\ a_2 e^{-t} \end{bmatrix}$$

Then the DE implies

$$\begin{bmatrix} -a_1 e^{-t} \\ -a_2 e^{-t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} a_1 e^{-t} \\ a_2 e^{-t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix}$$

We may cancel e^{-t} on both sides of the equation and then we have

$$\begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + 2 \\ -3a_1 + 5a_2 \end{bmatrix}$$

Thus

$$\begin{aligned} 2a_1 + a_2 &= -2 \\ 3a_1 - 6a_2 &= 0 \end{aligned}$$

We solve as usual.

$$\begin{bmatrix} 2 & 1 & -2 \\ 3 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 3 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & \frac{-15}{2} & 3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{-6}{15} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{-12}{15} \\ 0 & 1 & \frac{-6}{15} \end{bmatrix}$$

so $a_1 = -\frac{4}{5}$ and $a_2 = -\frac{2}{5}$. Finally

$$x_p(t) = \begin{bmatrix} -\frac{4}{5}e^{-t} \\ -\frac{6}{15}e^{-t} \end{bmatrix}$$

Check on x_p

$$\begin{bmatrix} -\frac{4}{5}e^{-t} \\ -\frac{2}{5}e^{-t} \end{bmatrix}' = \begin{bmatrix} \frac{4}{5}e^{-t} \\ \frac{2}{5}e^{-t} \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -\frac{4}{5}e^{-t} \\ -\frac{6}{15}e^{-t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5}e^{-t} \\ \frac{2}{5}e^{-t} \end{bmatrix}$$

$$x(t) = x_h(t) + x_p(t) = c_1\begin{bmatrix} 1e^{2t} \\ 1e^{2t} \end{bmatrix} + c_2\begin{bmatrix} \frac{1}{3}e^{4t} \\ e^{4t} \end{bmatrix} + \begin{bmatrix} -\frac{4}{5}e^{-t} \\ -\frac{2}{5}e^{-t} \end{bmatrix}$$

3 [20 **pts**.] Find the inverse of the matrix

$$\begin{bmatrix} 1 & 4 & 4 \\ 1 & 5 & 4 \\ 1 & 4 & 5 \end{bmatrix}$$

Solution: We form

$$\begin{bmatrix} 1 & 4 & 4 & 1 & 0 & 0 \\ 1 & 5 & 4 & 0 & 1 & 0 \\ 1 & 4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

Multiply row 1 by –1 and add it to rows 2 and 3

$$\begin{bmatrix} 1 & 4 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Multiply row 3 by –4 and add it to row 1

$$\begin{bmatrix} 1 & 4 & 0 & 5 & 0 & -4 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Multiply row 2 by –4 and add it to row 1

$$\begin{bmatrix} 1 & 0 & 0 & 9 & -4 & -4 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Thus the inverse is

$$\begin{bmatrix} 9 & -4 & -4 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

SNB check: $\begin{bmatrix} 1 & 4 & 4 \\ 1 & 5 & 4 \\ 1 & 4 & 5 \end{bmatrix}$, inverse: $\begin{bmatrix} 9 & -4 & -4 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$