Ma 227 Exam I B Solutions 10/4/12

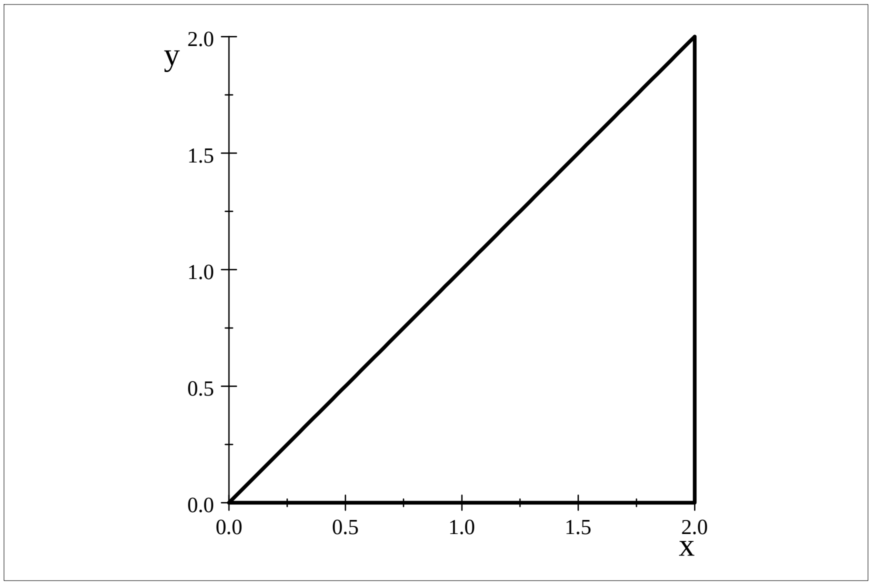
Name: ________________________________

Lecture Section: ______

1 [25 **pts**.] Evaluate

$$\int_0^2 \int_y^2 e^{x^2} dx dy$$

Sketch the region of integration.
Solution: The region of integration is shown below.



To evaluate the integral we reverse the order of integration.

$$\int_0^2 \int_y^2 e^{x^2} dx dy = \int_0^2 \int_0^x e^{x^2} dy dx$$

$$= \int_0^2 x e^{x^2} dx = \left. \frac{e^{x^2}}{2} \right|_0^2 = \frac{1}{2}\left[e^4 - 1\right]$$

2 a [20 **pts**.] Evaluate

$$\iint_R \left(x^2 + y^2\right) dA$$

where R is the circle $x^2 + y^2 = 2y$. Sketch R.
Solution: The equation of the circle may be put in standard form.

$$x^2 + y^2 - 2y = x^2 + (y-1)^2 - 1 = 0$$

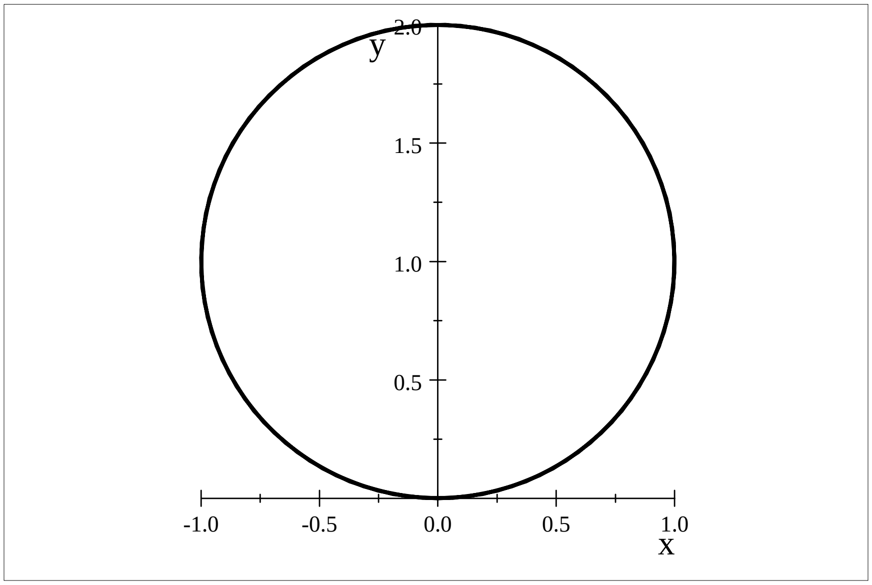
or

$$x^2 + (y-1)^2 = 1$$

This circle is centered at $(0,1)$ and has radius 1. In polar coordinates since $x^2+y^2=r^2$ and $y=r\sin\theta$, we have the eqaution

$$r = 2\sin\theta$$

$2\sin\theta$



Since the circle is only in the first and second quadrants, then

$$\begin{aligned}\iint_R \left(x^2+y^2\right)dA &= \int_0^{\pi}\int_0^{2\sin\theta} r^2 r\,dr\,d\theta \\ &= \int_0^{\pi} \frac{r^4}{4}\Big|_0^{2\sin\theta} d\theta \\ &= 4\int_0^{\pi} \sin^4\theta\,d\theta \\ &= 4\left[\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_0^{\pi} = \frac{3}{2}\pi\end{aligned}$$

2 b [15 **pts**.] Give an integral in polar coordinates for the surface area of the part of the hyperboloid $z=x^2-y^2$ that lies within the cylinder $x^2+y^2=1$. DO NOT EVALUATE THIS INTEGRAL.

Solution: Here $f(x,y)=x^2-y^2$ and $0\le x^2+y^2\le 1$. Thus $f_x=2x$, $f_y=2y$ and therefore

$$
\begin{aligned}
A(S) &= \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA \\
&= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA \\
&= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r dr d\theta
\end{aligned}
$$

3 [20 **pts**.] Use cylindrical coordinates to set up an iterated triple integral for the volume of the solid that lies within both the cylinder $x^2 + y^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 9$. DO NOT EVALUATE THIS INTEGRAL.

Solution: In cylindrical coordinates the volume E is the solid region within the cylinder $r = 2$ bounded above and below by the sphere $r^2 + z^2 = 9$. So

$$
E = \left\{ (r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, -\sqrt{9 - r^2} \le z \le \sqrt{9 - r^2} \right\}
$$

Hence the volume is given by

$$
\iiint_E dV = \int_0^{2\pi} \int_0^2 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r dz dr d\theta
$$

4 [20 **pts**.] Use spherical coordinates to evaluate

$$
\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2}\, dz dy dx
$$

Solution: The region of integration is the half of the sphere of radius 2 centered at the origin with $y \ge 0$.

$$
\begin{aligned}
&\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2}\, dz dy dx \\
&= \int_0^{\pi} \int_0^{\pi} \int_0^2 (\rho \sin\phi \sin\theta)^2 \sqrt{\rho^2}\, \rho^2 \sin\phi\, d\rho d\phi d\theta \\
&= \int_0^{\pi} \sin^2\theta d\theta \int_0^{\pi} \sin^3\phi d\phi \int_0^2 \rho^5 d\rho \\
&= \left[-\frac{1}{2}\cos\theta \sin\theta + \frac{1}{2}\theta \right]_0^{\pi} \left[-\frac{1}{3}\sin^2\theta \cos\theta - \frac{2}{3}\cos\theta \right]_0^{\pi} \left[\frac{\rho^6}{6} \right]_0^2 \\
&= \left[\frac{\pi}{2} \right] \left[\frac{2}{3} + \frac{2}{3} \right] \left[\frac{32}{3} \right] = \frac{64}{9}\pi
\end{aligned}
$$