Ma 227 Exam I B Solutions 10/3/13

Name: ________________________________

Lecture Section: ______

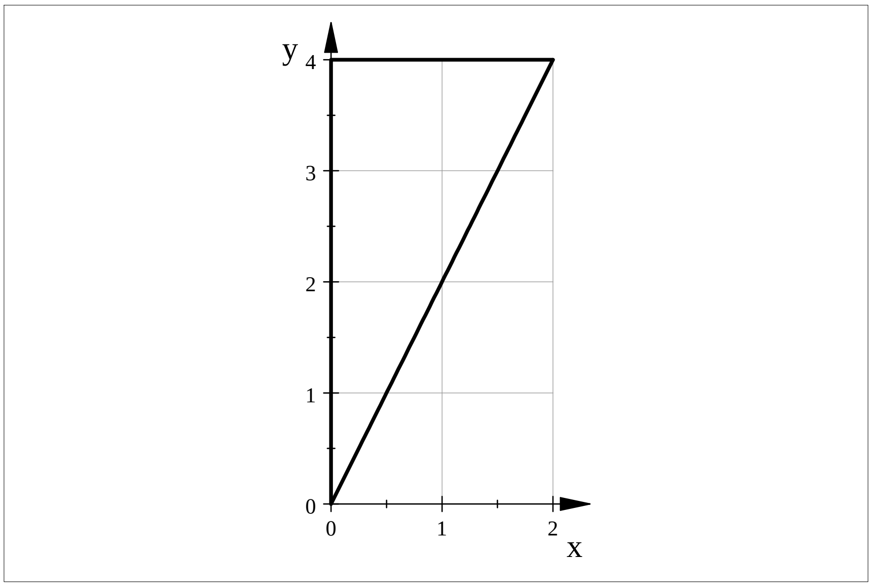
1 [25 **pts**.] Evaluate the integral

$$\iint_D e^{2y^2}\,dA$$

where D is the triangle bounded by the lines $y = 2x, y = 4,$ and $x = 0.$ Sketch $D.$

Solution: The lines $y = 2x, y = 4$ intersect at $(2,4).$ Thus D is shown below.

$\frac{x}{2}$



We cannot do the y integration first. Thus

$$\iint_D e^{2y^2}\,dA = \int_0^4 \int_0^{\frac{1}{2}y} e^{2y^2}\,dx\,dy$$

$$= \int_0^4 \frac{1}{2} y e^{2y^2}\,dy = \frac{1}{8} e^{2y^2}\Big|_0^4 = \frac{1}{8}\left(e^{32} - 1\right)$$

2 a [20 **pts**.] Calculate the

$$\iint_R \cos\sqrt{x^2 + y^2}\,dA$$

The region R is the disk $x^2 + y^2 \leq \pi^2.$

Solution: We switch to polar coordinates. Then we have (Using an integral from the table.)

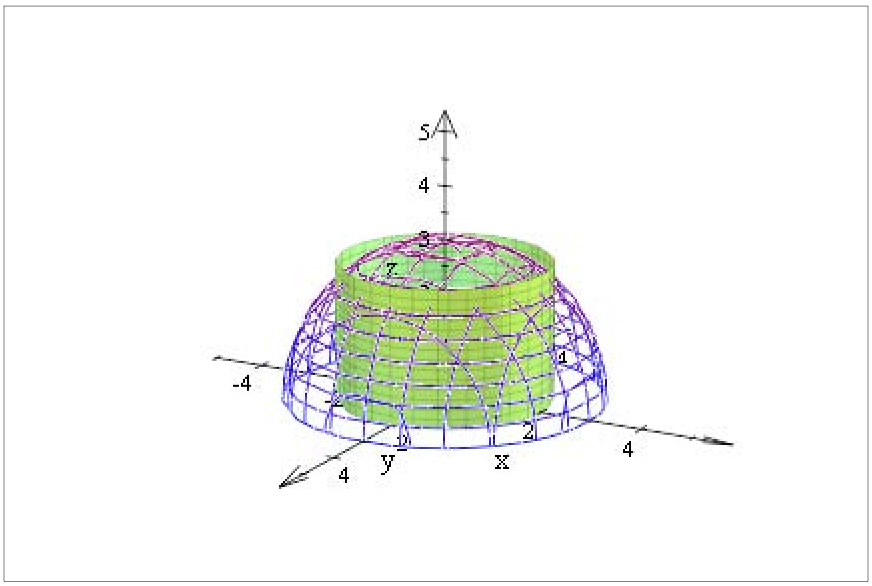
$$\iint_R \cos\sqrt{x^2+y^2}\,dA = \iint_R \cos\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\,r dr d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} r\cos r(r) dr d\theta = \int_0^{2\pi} [\cos r + r\sin r]_0^{\pi} d\theta$$
$$= \int_0^{2\pi} [-2] = -4\pi$$

2 b $\left[15\ \textbf{pts}.\right]$ Give an integral in rectangular coordinates for the surface area of the portion of the hemisphere

$x^2+y^2+z^2 = 9,\ z \geq 0$ that lies above the circular region $x^2+y^2 \leq 4$ in the x, y –plane. Sketch the surface area to be found. DO NOT EVALUATE THIS INTEGRAL.

Solution:

$x^2+y^2+z^2 = 9$



$$\text{Surface Area} = \iint_R \sqrt{1+(z_x)^2+(z_y)^2}\,dA$$

where R is the projection of the surface onto the x, y –plane. The equation of the hemisphere can be written as $z^2 = 25 - x^2 - y^2$. Differentiating we have $2zz_x = -2x$ or $z_x = -\frac{x}{z}$. Similarly, $z_y = -\frac{y}{z}$. Therefore

$$\sqrt{1+(z_x)^2+(z_y)^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \frac{1}{z}\sqrt{z^2+x^2+y^2} = \frac{3}{\sqrt{9-x^2-y^2}}$$

$$\text{Surface Area} = \iint_{x^2+y^2\leq 4} \frac{3}{\sqrt{9-x^2-y^2}}\,dA$$
$$= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{3}{\sqrt{9-x^2-y^2}}\,dxdy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{3}{\sqrt{9-x^2-y^2}}\,dydx$$

3 [25 **pts**.] Give a triple integral in cylindrical coordinates for the volume of the of the solid region cut from

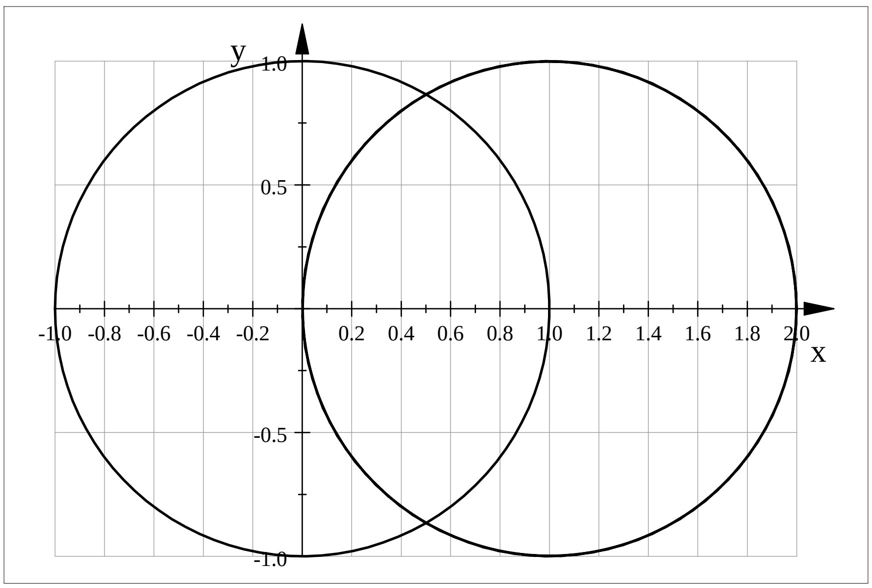
the sphere

$$x^2 + y^2 + z^2 = 1$$

by the cylinder $r = 2\cos\theta$. Sketch the solid. DO NOT EVALUATE THIS INTEGRAL.

Solution: In the x, y –plane $r = 2\cos\theta$ is the circle $(x-1)^2 + y^2 = 1$ shown below. We also need to plot the circle $r = 1$.

$2\sin\theta$



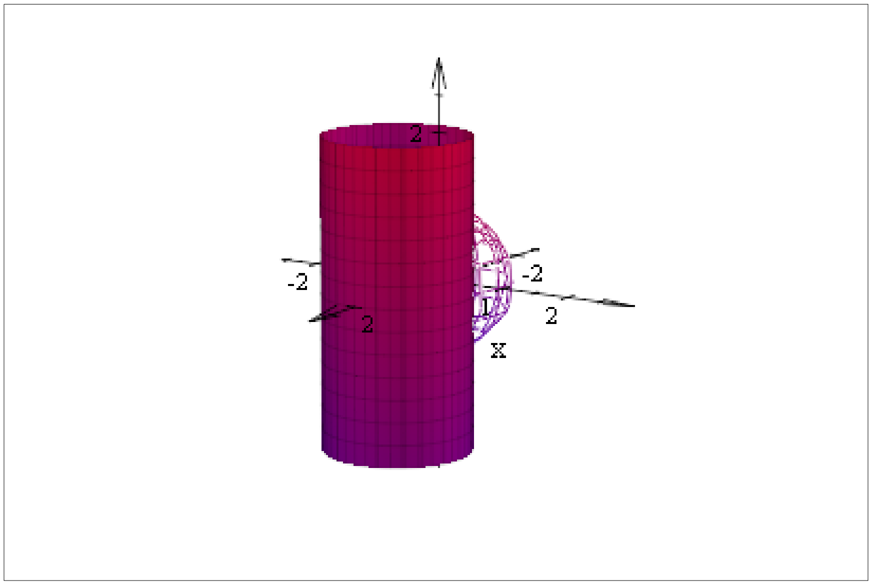
The circles intersect at

$$2\cos\theta = 1$$

or $\cos\theta = \frac{1}{2}$ that is at $\theta = \pm\frac{\pi}{3}$. Hence there are 3 areas of integration.

The volume is shown below.

$x^2 + y^2 + z^2 = 1$



We shall use cylindrical coordinates. In cylindrical coordinates the equation of the sphere is $r^2 + z^2 = 1$ so

$$-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$$

From the picture of the two circles shown in the x, y –plane above three integrals are required for the volume. The volume is given by

$$V = \int_{\frac{-\pi}{2}}^{-\frac{\pi}{3}} \int_0^{2\cos\theta} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr d\theta + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr d\theta$$

4 [20 **pts**.] Give an integral expression in **spherical** coordinates for the volume of the solid that lies above

the cone $\phi = \frac{\pi}{4}$ and below the sphere $\rho = 2a\cos\phi$. Sketch the volume. *Do not evaluate this expression.*

Solution: The equation of the sphere may be written as $\rho^2 = 2a\rho\cos\phi$ which in Cartesian coordinates is

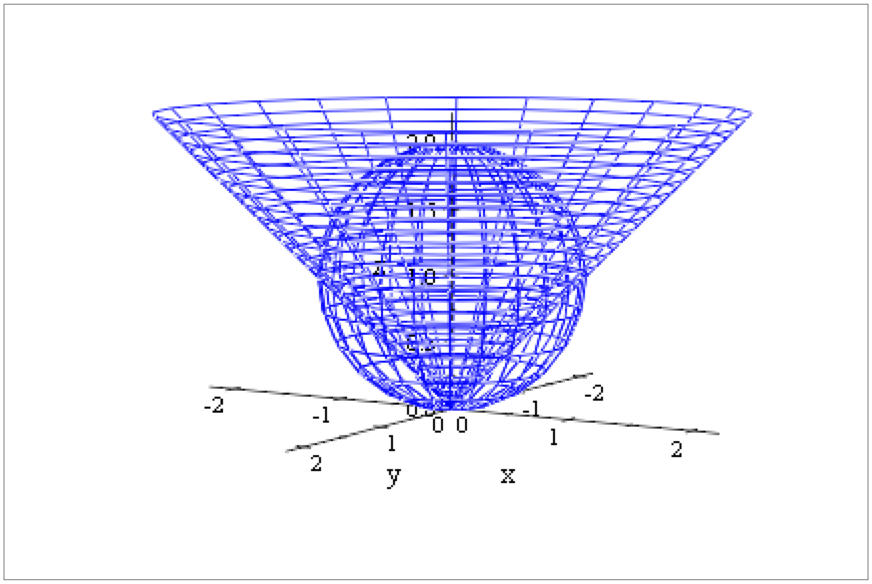
$$x^2 + y^2 + z^2 = 2az$$

or

$$x^2 + y^2 + (z-a)^2 = a^2$$

Thus the sphere has center at $(0, 0, a)$ and radius a.

$\left(\rho, \theta, \frac{\pi}{4}\right)$



The volume is given by

$$\text{Volume} = \iiint_V dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2a\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Table of Integrals

$\int x\sin x dx = \sin x - x\cos x + C$
$\int x\cos x dx = \cos x + x\sin x + C$
$\int \sin^2 x dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + C$
$\int \sin^4 x dx = \frac{3}{8}x - \frac{3}{16}\pi - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$
$\int \cos^4 x dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$
$\int te^{at}dt = \frac{1}{a^2}e^{at}(at - 1) + C$
$\int t^2e^{at}dt = \frac{1}{a^3}e^{at}\left(a^2t^2 - 2at + 2\right) + C$