**Ma 227** **Exam II Solutions** **11/7/07**

**Name**: ________________________________

**Lecture Section**: ______

*I pledge my honor that I have abided by the Stevens Honor System.* ___________________________________

**You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.**
**There is a table of integrals on the last page of the exam.**

Score on Problem #1 ________

#2a ________

#2b ________

#3 ________

#4 ________

Total Score ________

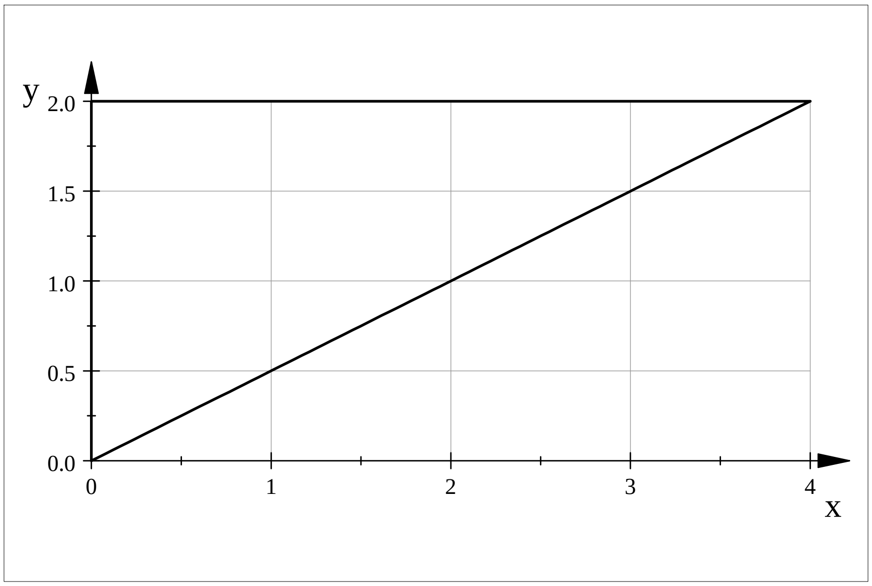
**1** [25 **pts**.] Let $R$ be the region in the $x,y$ –plane bounded by the $y$ –axis, the line $x = 2y$ and the line $y = 2.$ Sketch $R,$ give two expressions for

$$\iint_R \sqrt{4-y^2}\,dA$$

and evaluate one of them.

Solution: The line $x = 2y$ intersects the line $y = 2$ when $x = 4,$ that is at the point $(4,2).$ The region $R$ is the triangle shown below.

$\frac{x}{2}$




Therefore

$$\iint_R \sqrt{4-y^2}\,dA = \int_0^2 \int_0^{2y} \sqrt{4-y^2}\,dxdy = \int_0^4 \int_{\frac{x}{2}}^2 \sqrt{4-y^2}\,dydx$$

We evaluate the first integral, since it is much easier than the second.

$$\begin{aligned}\int_0^2 \int_0^{2y} \sqrt{4-y^2}\,dxdy &= \int_0^2 \left[ x\sqrt{4-y^2} \right]_{x-0}^{x=2y} dy \\ &= \int_0^2 2y\left(4-y^2\right)^{\frac{1}{2}} dy \\ &= \left[ -\frac{2}{3}\left(4-y^2\right)^{\frac{3}{2}} \right]_0^2 = 0 + \frac{16}{3}\end{aligned}$$

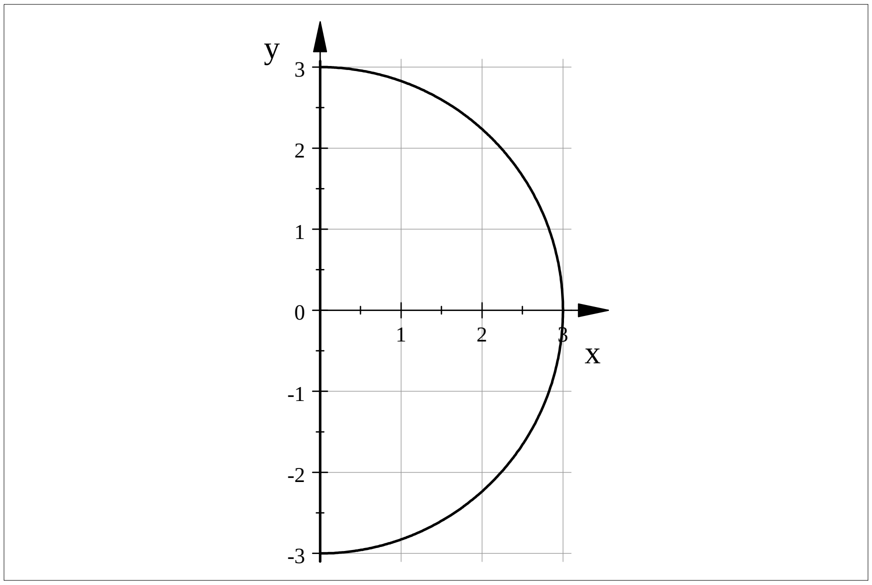
**2 a** $\left[\right.$20 **pts**.$\left.\right]$ Evaluate

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(x^2+y^2\right)^{\frac{5}{2}} dydx$$

Be sure to sketch the region of integration.

Solution: $y$ goes from $\sqrt{9-x^2}$ to -$\sqrt{9-x^2}$. Thus $y$ lies on the circle $x^2 + y^2 = 9.$ Since $x$ goes from 0 to $3,$ the region is only the part of the circle of radius 3 centered at the origin to the right of the $y$ –axis. Thus the region is

$x^2 + y^2 = 9$




Changing to polar coordinates we have

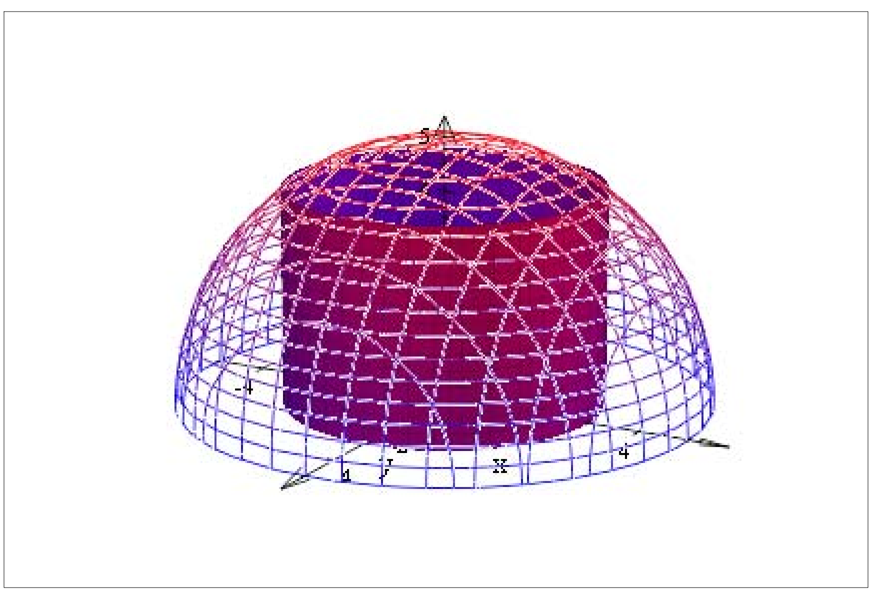
$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(x^2+y^2\right)^{\frac{5}{2}} dydx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 \left(r^2\right)^{\frac{5}{2}} rdrd\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^7}{7}\right]_0^3 d\theta = \frac{3^7}{7}\pi$$

**2 b** $\left[\right.$15 **pts**.$\left.\right]$ Give an integral in rectangular coordinates for the surface area of the portion of the hemisphere $x^2 + y^2 + z^2 = 25,\ z \geq 0$ that lies above the circular region $x^2 + y^2 \leq 9$ in the $x, y$ –plane. Do not evaluate this integral. Sketch the surface area to be found.

Solution:

$x^2 + y^2 + z^2 = 25$



$$\text{Surface Area} = \iint_R \sqrt{1+(z_x)^2+(z_y)^2}\,dA$$

where $R$ is the projection of the surface onto the $x,y$ –plane. The equation of the hemisphere can be written as $z^2 = 25 - x^2 - y^2$. Differentiating we have $2zz_x = -2x$ or $z_x = -\frac{x}{z}$. Similarly, $z_y = -\frac{y}{z}$. Therefore

$$\sqrt{1+(z_x)^2+(z_y)^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \frac{1}{z}\sqrt{z^2+x^2+y^2} = \frac{5}{\sqrt{25-x^2-y^2}}$$

$$\text{Surface Area} = \iint_{x^2+y^2\le 9} \frac{5}{\sqrt{25-x^2-y^2}}\,dA$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{5}{\sqrt{25-x^2-y^2}}\,dxdy = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25-x^2-y^2}}\,dydx$$

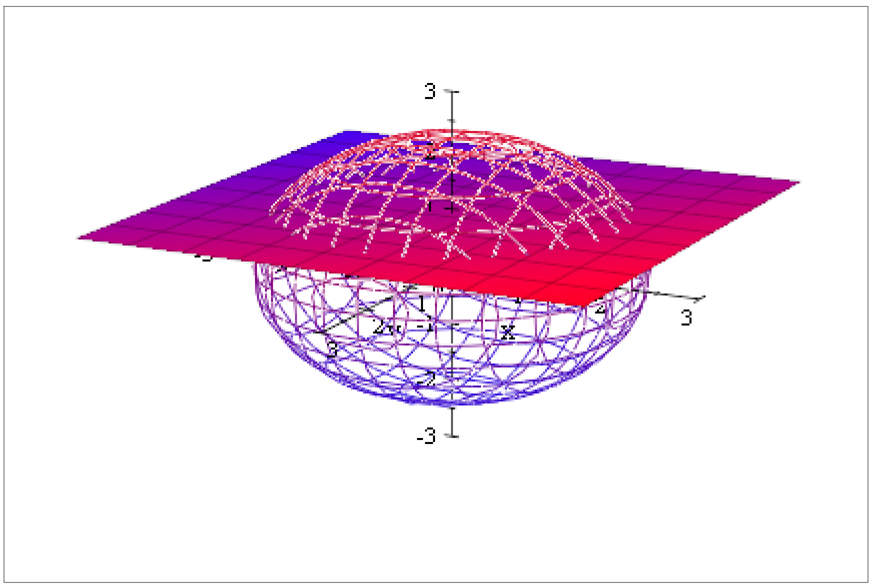
**3** [20 **pts**.] Set up, but DO NOT INTEGRATE, a triple integral to find the volume of the solid bounded above by $x^2+y^2+z^2 = 5$ and below by $z = 1$

using **spherical** coordinates.

Solution: This example was worked out on a handout that was distributed in lecture.

The region of integration is shown below. One uses Plot 3D, Implicit to get the picture.

$x^2+y^2+z^2 = 5$




$\rho$ will go from the plane $z = 1$ to the sphere $x^2+y^2+z^2 = 5$.

In spherical, $x^2+y^2+z^2 = 5 \Rightarrow \rho = \sqrt{5}$

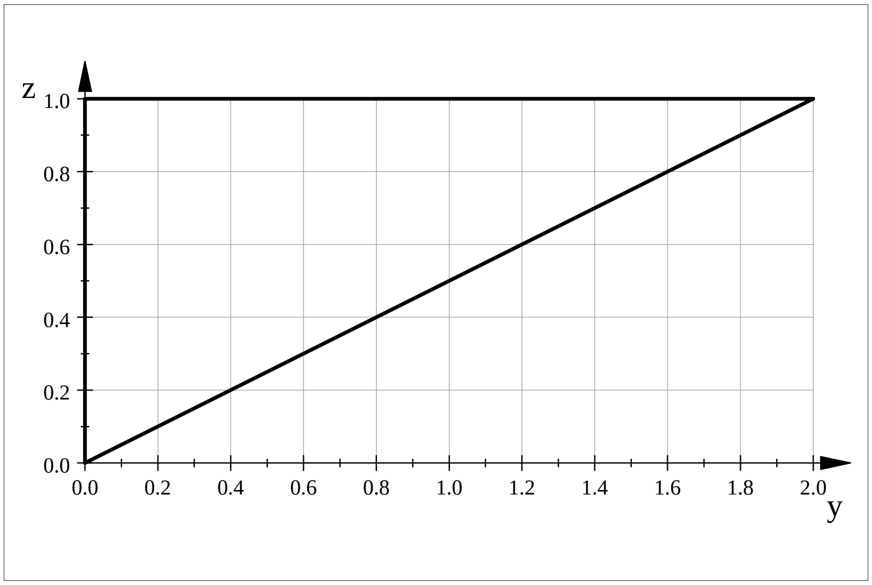
Also, $z = 1 \Rightarrow \rho\cos\phi = 1 \Rightarrow \rho = \sec\phi$.

So, $\sec\phi \le \rho \le \sqrt{5}$.

For $\phi$, we can form a right triangle with hypotenuse $\sqrt{5}$ (the radius of the sphere) and vertical side 1 which is the distance from the origin to $z = 1$. So the horizontal side is 2.

$\sqrt{5} = 2.2361$

$x$




Therefore, $\tan\phi = 2 \Rightarrow \phi = \arctan 2$.

So, $0 \le \phi \le \arctan 2$.

The volume is:

$$V = \int_0^{2\pi}\int_0^{\arctan 2}\int_{\sec\phi}^{\sqrt{5}} p^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

**4** $[20$ **pts**.$]$ Give an integral expression in **cylindrical** coordinates for

$$\iiint_V y^2 dV$$

where $V$ is the part of the ellipsoid $4x^2 + 4y^2 + z^2 = 16$ lying above the $xy$ –plane. *Do not evaluate this expression.*

Solution: $z = \sqrt{16 - 4x^2 - 4y^2} = \sqrt{16 - 4r^2}$. The ellipsoid intersects the $x,y$ –plane when $z = 0$, that is for $x^2 + y^2 = 4$. The equation of this circle is $r = 2$. Thus

$$\iiint_V y^2 dV = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-4r^2}} \left(r^2\right)\sin^2\theta(r)\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-4r^2}} \left(r^3\right)\sin^2\theta \, dz\,dr\,d\theta$$

# Table of Integrals

| |
|---|
| $\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$ |
| $\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$ |
| $\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$ |
| $\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$ |
| |