Ma 227 **Exam II Solutions** **11/9/09**

Name: ________________________________

Lecture Section: ______

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.
There is a table of integrals on the last page of the exam.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

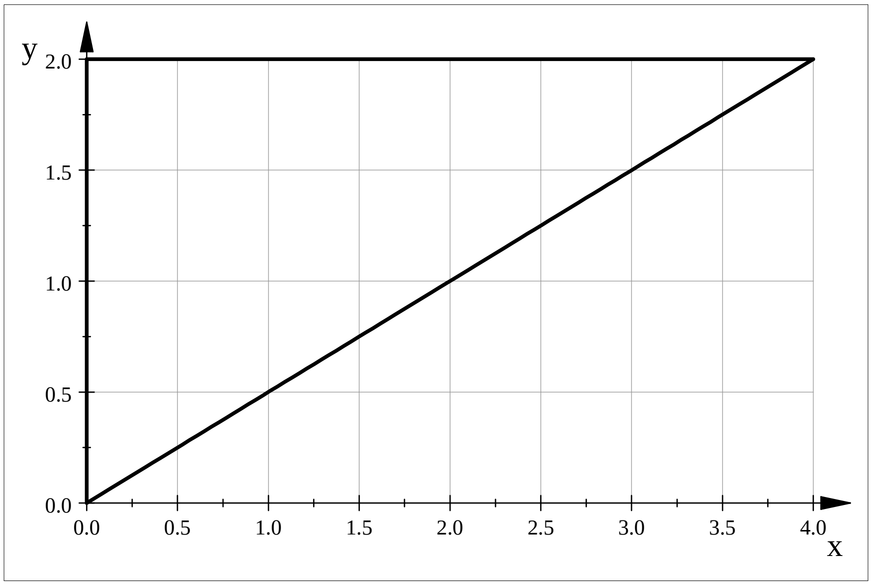
#5 ________

Total Score ________

1 [20 **pts**.] Evaluate

$$\iint_R e^{y^2} dA$$

where R is the region shown below.



The vertices of the triangle are $(0,0),(0,2),$ and $(4,2)$. The hypotenuse of the triangle is therefore the line $y = \frac{x}{2}$. This line intersects the line $y = 2$ at $(4,2)$We can write the integral as

$$\iint_R e^{y^2} dydx$$

However, there is not way to evaluate the integral of e^{y^2}. Thus we consider

$$\iint_R e^{y^2} dxdy = \int_0^2 \int_0^{2y} e^{y^2} dxdy$$

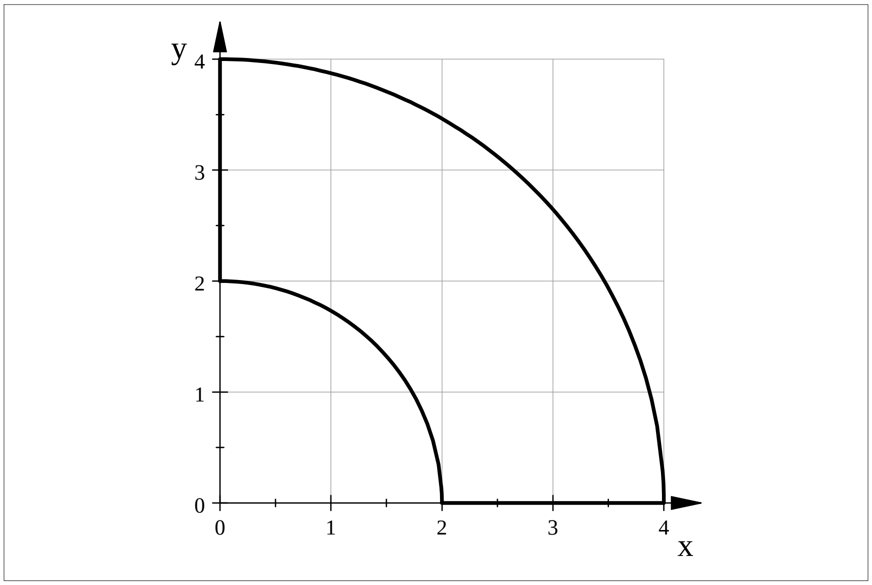
This integral we can evaluate.

$$\int_0^2 \int_0^{2y} e^{y^2} dxdy = \int_0^2 2ye^{y^2} dy = e^{y^2} \Big|_0^2 = e^4 - 1$$

2 [20 **pts**.] Evaluate

$$\iint_R (x+y)\,dA$$

where R is the quarter annulus shown below.



Solution: The quarter annulus is given by

$$4 \le x^2 + y^2 \le 16, \quad x, y \ge 0$$

Given the geometry of the region, we shall use polar coordinates. In polar coordinates the region is

$$2 \le r \le 4, \quad 0 \le \theta \le \frac{\pi}{2}$$

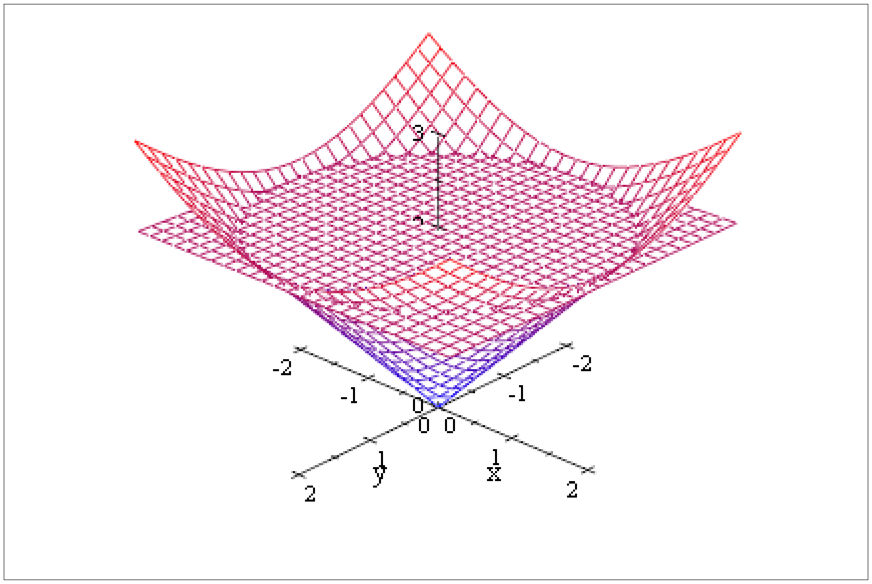
Therefore

$$\begin{aligned}
\iint_R (x+y)\,dA &= \int_0^{\frac{\pi}{2}} \int_2^4 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{r^3}{3}\Big|_2^4 (\cos\theta + \sin\theta)\,d\theta \\
&= \left(\frac{4^3}{3} - \frac{2^3}{3}\right)(\sin\theta - \cos\theta)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{2^3}{3}\left(2^3 - 1\right)(1+1) = \frac{8}{3}(7)(2) = \frac{112}{3}
\end{aligned}$$

3 $\left[20 \textbf{ pts.}\right]$ Give an integral in rectangular coordinates for the surface area of the portion of the cone $z^2 = x^2 + y^2$ that lies above the circular region $x^2 + y^2 \leq 4$ in the x, y –plane and then find the surface area. Sketch the surface area to be found.

Solution: The surface area required is the part of the cone below the plane $z = 2$.

$\sqrt{x^2 + y^2}$



$$\text{Surface Area} = \iint\limits_R \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA$$

Differentiating we have $2zz_x = 2x$ or $z_x = \frac{x}{z}$. Similarly, $z_y = \frac{y}{z}$. Therefore

$$\sqrt{1 + (z_x)^2 + (z_y)^2} = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} = \frac{1}{z}\sqrt{z^2 + x^2 + y^2} = \frac{1}{z}\sqrt{z^2 + z^2} = \sqrt{2}$$

Thus

$$\text{Surface Area} = \iint\limits_{x^2+y^2\leq 4} \sqrt{2}\, dA$$

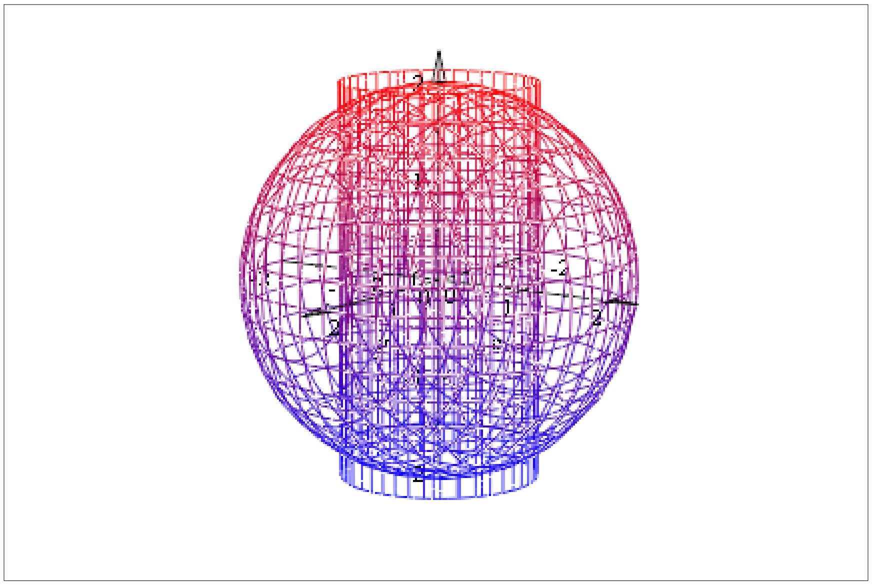
We switch to polar coordinates to evaluate this integral

$$\text{Surface Area} = \iint\limits_{x^2+y^2\leq 4} \sqrt{2}\, dA = \sqrt{2}\int_2^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dxdy = \sqrt{2}\int_2^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dydx$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^2 r dr d\theta = 2\sqrt{2}\int_0^{2\pi} d\theta = 4\sqrt{2}\,\pi$$

Instead of actually doing the evaluation, one could note that $\iint\limits_{x^2+y^2\leq 4} \sqrt{2}\, dA = \sqrt{2}$ times the area of a circle of radius 2 and hence obtain the answer above.

4 $\left[20\ \textbf{pts}.\right]$ Give a triple integral in **cylindrical** coordinates for the volume that lies both within the cylinder $x^2+y^2=1$ and the sphere
$x^2+y^2+z^2=4$. DO NOT EVALUATE THIS INTEGRAL. Sketch the volume in question.
Solution: The volume is shown below.
$x^2+y^2+z^2=4$



The volume V is the solid region within the cylinder $r=1$ bounded above and below by the $r^2+z^2=4$. Hence $-\sqrt{4-r^2}\leq z\leq\sqrt{4-r^2}, 0\leq r\leq 1,$ and $0\leq\theta\leq 2\pi$.

$$\text{Volume}=\iiint_V dV=\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$=8\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \text{ by symmetry}$$

5 [20 **pts**.] Evaluate

$$\iiint_V e^{\sqrt{x^2+y^2+z^2}}\, dV$$

where V is the part of the sphere $x^2+y^2+z^2=9$ in the first octant.

Solution: We use spherical coordinates The euquation of the sphere is $\rho=3$. Since we are in the first octant, then $0 \le \theta \le \frac{\pi}{2}$, and $0 \le \phi \le \frac{\pi}{2}$. Hence,

$$\begin{aligned}
\iiint_V e^{\sqrt{x^2+y^2+z^2}}\, dV &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^2 \sin\phi e^{\sqrt{\rho^2}}\, d\rho d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^2 \sin\phi e^{\rho} d\rho d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(e^{\rho}\left(\rho^2-2\rho+2\right)\right)\Big|_0^3 \sin\phi d\phi d\theta \\
&= \left(5e^3-2\right) \int_0^{\frac{\pi}{2}} (-\cos\phi)|_0^{\frac{\pi}{2}}\, d\theta \\
&= \left(5e^3-2\right)(0+1) \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}\left(5e^3-2\right)
\end{aligned}$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int t e^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$