Ma 227 **Exam IIA Solutions** **4/3/06**

Name: ________________________________

Lecture Section: ______

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

There is a table of integrals on the last page of the exam.

Score on Problem #1 ________

#2 ________

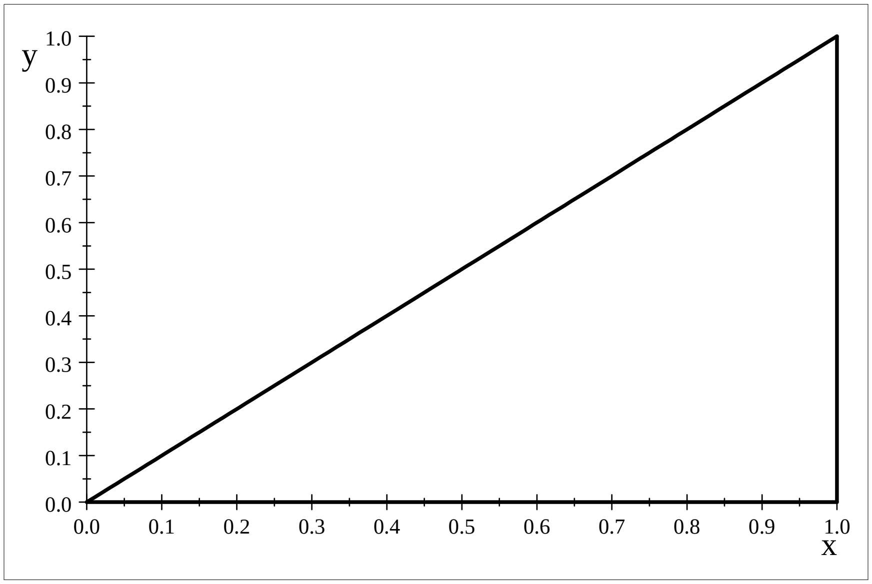
#3 ________

#4 ________

Total Score ________

1 [20 **pts**.] Find the volume of the prism whose base is the triangle in the x, y –plane bounded by the x –axis, and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = 3 - x - y$. Be sure to sketch the triangle in the x, y –plane.

x



Thus

$$V = \iint\limits_{\text{Triangle}} (3 - x - y)dA = \int_0^1 \int_0^x (3 - x - y)dydx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left[3x - \frac{3}{2}x^2 \right] dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_0^1 = 1$$

or

$$V = \iint\limits_{\text{Triangle}} (3 - x - y)dA = \int_0^1 \int_y^1 (3 - x - y)dxdy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left[3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right] dy = \int_0^1 \left[\frac{5}{2} - 4y + \frac{3y^2}{2} \right] dy = \left[\frac{5y}{2} - 2y^2 + \frac{y^3}{2} \right]_0^1 = 1$$

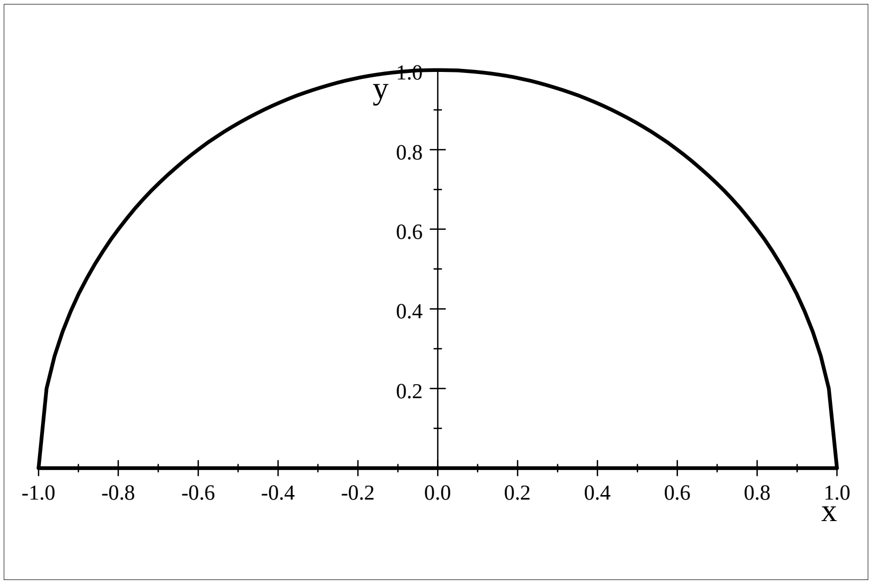
2 a $\left[\right.$20 **pts**.$\left.\right]$ Evaluate the iterated integral

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} x^2 \left(x^2 + y^2\right)^{\frac{1}{2}} dydx$$

Be sure to sketch the region of integration in the x, y –plane.

Solution: y goes from 0 to $\sqrt{1 - x^2}$. This means it goes from 0 to the circle $x^2 + y^2 = 1$. x goes from -1 to 1. Thus the region of integration is the upper half of the unit circle.

$\sqrt{1 - x^2}$



To evaluate the integral we switch to polar coordinates. Then

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2\left(x^2+y^2\right)^{\frac{1}{2}} dydx = \int_0^{\pi}\int_0^1\left(r^2\cos^2\theta\right)r(r)drd\theta$$

$$= \int_0^{\pi}\int_0^1 r^4\cos^2\theta d\theta = \frac{1}{5}\int_0^{\pi}\cos^2\theta d\theta$$

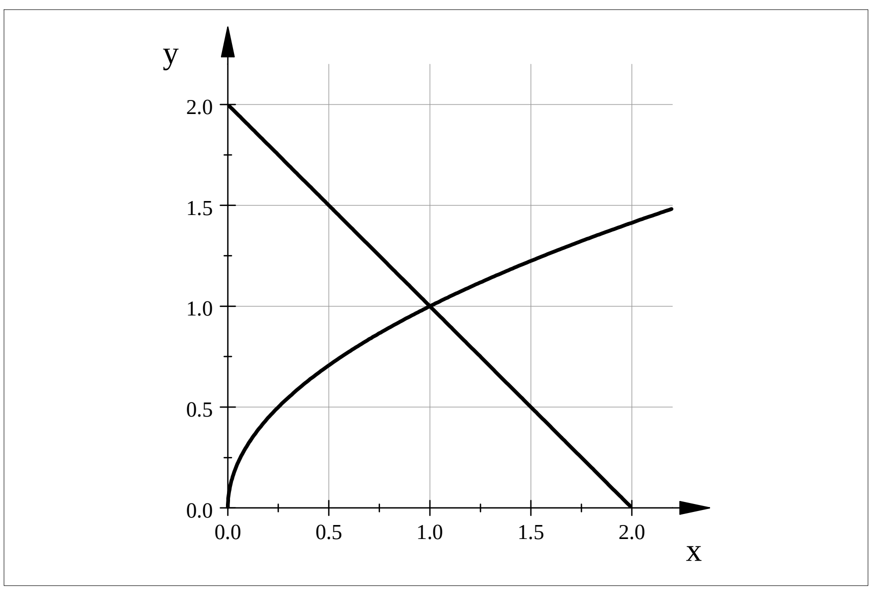
$$= \frac{1}{5}\left[\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right]_0^{\pi} = \frac{\pi}{10}$$

2 b [20 **pts**.] Sketch the region of integration and change the order of integration for the integral

$$\int_0^1\int_{y^2}^{2-y} f(x,y)dxdy$$

Solution: x goes from the parabola $x = y^2$ to the line $x = 2 - y$ or $y = 2 - x$. The region of integration is shown below

$x = y^2$



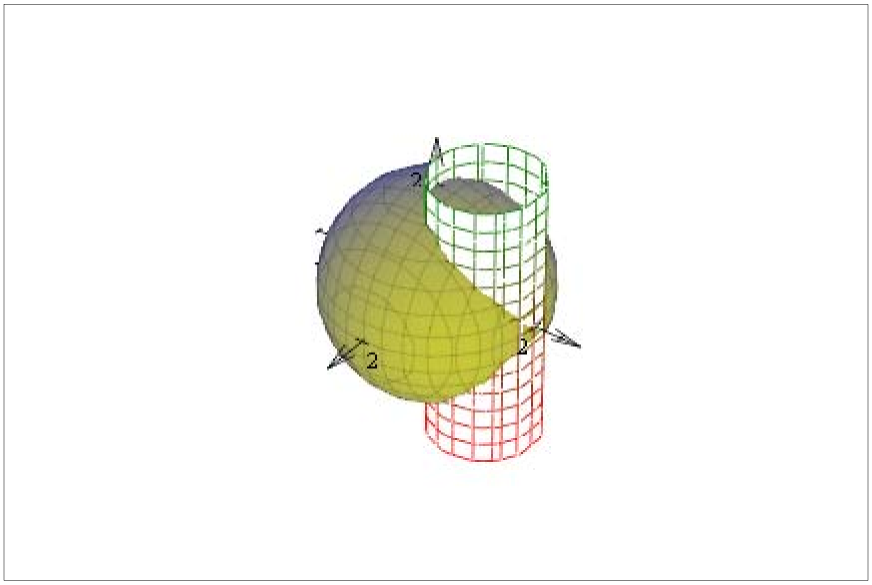
The two curves intersect when $(2-x)^2 = x$ or when $x^2 - 5x + 4 = 0$. That is when $(x-4)(x-1) = 0$. This gives $x = 1$ and $x = 4$. However, we reject $x = 4$, since it is out of the range of the region. Thus the point of intersection is $(1,1)$. Two integrals are needed to do the y integration first. Thus we have

$$\int_0^1 \int_{y^2}^{2-y} f(x,y)dxdy = \int_0^1 \int_0^{\sqrt{x}} f(x,y)dydx + \int_1^2 \int_0^{2-x} f(x,y)dydx$$

3 [20 **pts**.] Let V be the volume cut out of the sphere $x^2 + y^2 + z^2 = 4$ by the cylinder $x^2 + y^2 = 2y$. Give a triple integral expression in **cylindrical** coordinates for the volume of V. Sketch V. *Do not evaluate the integral you give.*

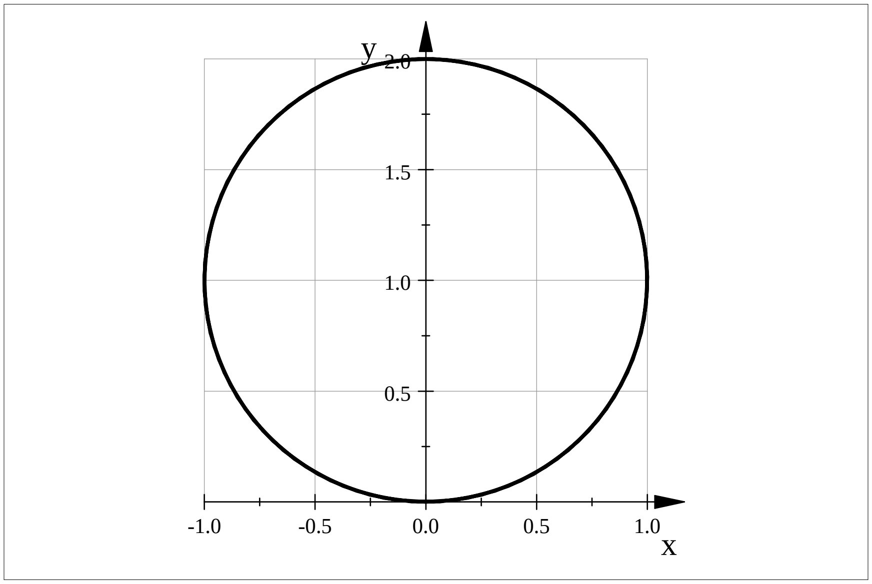
Solution: The region V is shown below. The sphere and the cylinder were plotted using Plot 3D, Implicit.

$x^2 + y^2 + z^2 = 4$



The graph of the cylinder in polar coordinates in the x,y –plane is given by $r^2 = 2r\sin\theta$ or $r = 2\sin\theta$. Graphing this in polar yields

$2\sin\theta$



Thus the region of integration in the x, y –plane is over the circle of radius 2 centered at $(0, 1)$. If we find the volume in the first octant and then double it, then z goes from the x, y –plane to the sphere. The equation of the sphere in cylindrical coordinates is $z^2 = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$. Therefore

$$V = 2\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

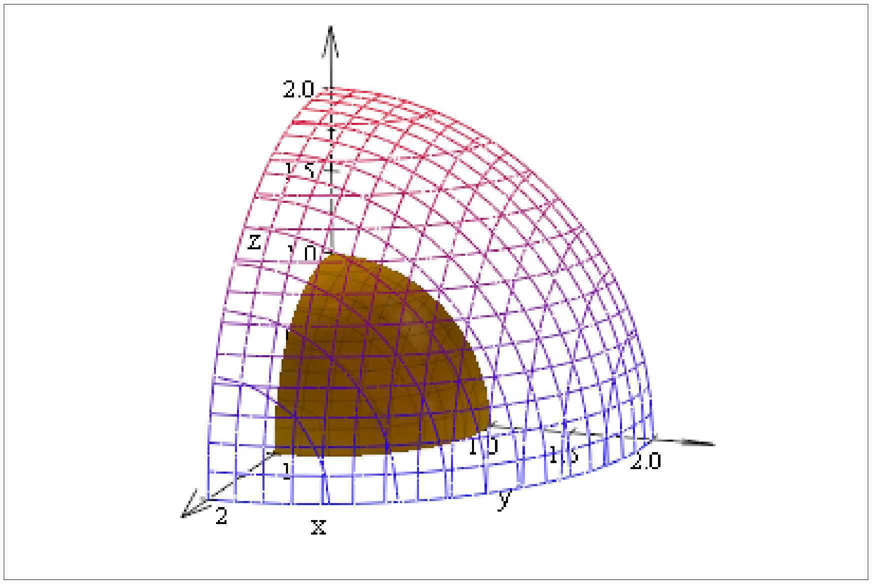
4 [20 **pts**.] Give an integral expression in **spherical** coordinates for

$$\iiint_V x e^{(x^2+y^2+z^2)^2} dV$$

where V is the solid that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the *first octant*. *Do not evaluate this expression.*

Solution: The two surfaces have radii of 1 and 2. Thus their equations are $\rho = 1$ and $\rho = 2$. They are shown below. Plot 3D, Implicit was used.

$x^2 + y^2 + z^2 = 1$



Since we are in the first octant θ and ϕ both go from 0 to $\frac{\pi}{2}$. Therefore, since $\rho^2 = x^2 + y^2 + z^2$

$$\iiint\limits_V xe^{(x^2+y^2+z^2)^2} dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 \rho \cos\theta \sin\phi e^{(\rho^2)^2} \rho^2 \sin\phi d\rho d\phi d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 \rho \cos\theta \sin\phi e^{\rho^4} \rho^2 \sin\phi d\rho d\phi d\theta$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$