Ma 227 **Exam II A Solutions** **11/13/12**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#2 ________

#3a ________

#3b ________

#4 ________

Total Score ________

1 a [20 **pts**.] Evaluate

$$\int_C xy^4 ds$$

where C is the right half of the circle, $x^2 + y^2 = 16$ traversed in the counter clockwise direction.
Solution: We first need a parameterization of the circle. This is given by,

$$x = 4\cos t \quad y = 4\sin t$$

The range of t that gives the right half of the circle traversed in the counter clockwise direction is

$$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\frac{dx}{dt} = -4\sin t \quad \frac{dy}{dt} = 4\cos t$$

Thus

$$\int_C xy^4 ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(4\cos t(4\sin t)^4\right)\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt$$

$$= 4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\left(\sin^4 t\right)dt = 4^6 \frac{\sin^5 t}{5}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{4^6}{5}(1-(-1)) = \frac{2\left(4^6\right)}{5} = \frac{8192}{5}$$

1b [**15 pts**.] Let

$$\vec{F}(x,y,z) = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$$

Show that

$$\mathrm{div}\left(\mathrm{curl}\vec{F}\right) = \nabla \cdot \left(\nabla \times \vec{F}\right) = 0$$

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{matrix}$$

$$= \frac{\partial F_3}{\partial y}\vec{i} + \frac{\partial F_1}{\partial z}\vec{j} + \frac{\partial F_2}{\partial x}\vec{k} - \frac{\partial F_1}{\partial y}\vec{k} - \frac{\partial F_2}{\partial z}\vec{i} - \frac{\partial F_3}{\partial x}\vec{j}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Therefore

$$\nabla \cdot \left(\nabla \times \vec{F}\right) = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0$$

2 [20 **pts**.] Find a function $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}(x,y,z) = \left(2x\cos y - 2z^3\right)\vec{i} + \left(2ye^z - x^2\sin y\right)\vec{j} + \left(3 + y^2e^z - 6xz^2\right)\vec{k}$$

Solution: Not part of the solution. Note that $\nabla \times \left(2x\cos y - 2z^3, 2ye^z - x^2\sin y, 3 + y^2e^z - 6xz^2\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so this force field is conservative and such a $\phi(x,y,z)$ exists.

$$\phi_x = 2x\cos y - 2z^3$$

so integrating with respect to x and holding y and z fixed, we have

$$\phi(x,y,z) = x^2\cos y - 2xz^3 + g(y,z)$$

Differentiating with respect to y we have

$$\phi_y = -x^2\sin y + g_y = 2ye^z - x^2\sin y$$

and

$$g_y = 2ye^z$$

Integrating with respect to y while holding z fixed we have

$$g = y^2e^z + h(z)$$

Thus

$$\phi(x,y,z) = x^2\cos y - 2xz^3 + y^2e^z + h(z)$$

Differentiating with respect to z we have

$$\phi_z = -6xz^2 + y^2e^z + h'(z) = 3 + y^2e^z - 6xz^2$$

Therefore $h'(z) = 3$ and $h(z) = 3z + K$, where K is a constant. Hence finally we have

$$\phi(x,y,z) = x^2\cos y - 2xz^3 + y^2e^z + 3z + K$$

Any value of K may be chosen, since only one function is required.

3a [15 **pts**.] Evaluate

$$\oint_C y^3dx - x^3dy$$

directly without using Green's Theorem, where C is the positively oriented circle of radius 2 centered at the origin.

Solution: Let

$$x = 2\cos t, \quad y = 2\sin t \text{ where } \quad 0 \le t \le 2\pi$$

Then

$$\oint_C y^3dx - x^3dy = \int_0^{2\pi} \left(8\sin^3 t(-2\sin t) - 8\cos^3 t(2\cos t)\right)dt$$
$$= -16\int_0^{2\pi} \left(\sin^4 t + \cos^4 t\right)dt$$
$$= -16\left(\frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + \frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t\right)_0^{2\pi}$$
$$= -16\left(\frac{3}{4}\right)(2\pi) = -24\pi$$

3b [15 **pts**.] Evaluate the line integral in 3a, namely

$$\oint_C y^3dx - x^3dy$$

using Green's Theorem.
Solution: Here

$$P = y^3 \text{ and } Q = -x^3$$

Thus

$$P_y = 3y^2 \text{ and } Q_x = -3x^2$$

Hence

$$\oint_C y^3dx - x^3dy = \iint_{x^2+y^2\le 4} \left(-3x^2 - 3y^2\right)dA$$
$$= -3\int_0^{2\pi}\int_0^2 r^2 r\,dr\,d\theta$$
$$= -3\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta = -12\int_0^{2\pi} d\theta = -24\pi$$

4 [15 **pts**.] Use Green's Theorem to find the area of the ellipse bounded by

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Solution:

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

We parametrize the curve using

$$x = 2\cos t \quad y = 3\sin t \quad 0 \le t \le 2\pi$$

Hence

$$A = \frac{1}{2}\int_0^{2\pi} (2\cos t(3\cos t) - 3\sin t(-2\sin t))dt$$
$$= \frac{1}{2}\int_0^{2\pi} 6\left(\cos^2 t + \sin^2 t\right)dt = \frac{6}{2}\int_0^{2\pi} dt = 6\pi$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int \sin^4 t dt = \frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C$
$\int \cos^4 t dt = \frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$