Ma 227 **Exam II A Solutions** **11/5/13**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

1a [20 **pts**.] Evaluate

$$\int_C \vec{F} \bullet d\vec{r}$$

where $\vec{F} = z\vec{i} + y^2\vec{j} + x\vec{k}$ and C is the curve given by

$$C: \ x(t) = t+1, y(t) = e^t, z(t) = t^2 \quad 0 \le t \le 2$$

Solution:

$$\vec{F}(t) = t^2\vec{i} + e^{2t}\vec{j} + (t+1)\vec{k}$$

and

$$\vec{r}(t) = (t+1)\vec{i} + e^t\vec{j} + t^2\vec{k}$$

so

$$\vec{r}'(t) = \vec{i} + e^t\vec{j} + 2t\vec{k}$$

Thus

$$\begin{aligned}\int_C \vec{F} \bullet d\vec{r} &= \int_0^2 \vec{F}(t) \bullet \vec{r}'(t)dt \\ &= \int_0^2 \left(t^2 + e^{3t} + 2t^2 + 2t\right)dt \\ &= \int_0^2 \left(3t^2 + 2t + e^{3t}\right)dt \\ &= \left[t^3 + t^2 + \frac{1}{3}e^{3t}\right]_0^2 = 8 + 4 + \frac{1}{3}e^6 - \frac{1}{3} = \frac{1}{3}\left(35 + e^6\right)\end{aligned}$$

1b [**15 pts**.] If the function $f(x,y,z)$ has continuous second-order partial derivatives, show that

$$\text{curl}\left(\text{grad } f\right) = 0$$

Solution: $\nabla f = \text{grad } f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ so

$$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{matrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{k} = \vec{0}$$

2 [20 **pts**.] Find a function $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}(x,y,z) = \left(2xyz^{-1}\right)\vec{i} + \left(z + x^2z^{-1}\right)\vec{j} + \left(y - x^2yz^{-2}\right)\vec{k}$$

Solution:

$$\phi_x = 2xyz^{-1} \quad \phi_y = z + x^2z^{-1} \quad \phi_z = y - x^2yz^{-2}$$

Starting with ϕ_x and integrating with respect to x we have

$$\phi(x,y,z) = x^2yz^{-1} + g(y,z)$$

Hence

$$\phi_y = x^2z^{-1} + g_y = z + x^2z^{-1}$$

Therefore

$$g(y,z) = zy + h(z)$$

and

$$\phi(x,y,z) = x^2yz^{-1} + zy + h(z)$$

Therefore

$$\phi_z = -x^2yz^{-2} + y + h'(z) = y - x^2yz^{-2}$$

Hence $h(z) = K$ where K is a constant. Finally we have

$$\phi(x,y,z) = x^2yz^{-1} + zy + K$$

3a $\left[15\ \textbf{pts.}\right]$ Evaluate

$$\oint_C xy^2dx + xdy$$

directly without using Green's Theorem, where C is the positively oriented circle of radius 1 centered at the origin.

Solution: C may be parametrized as $x = \cos t, y = \sin t \quad 0 \leq t \leq 2\pi$ so $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j}$ and $\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j}$. We let

$$\vec{F}(x,y) = xy^2\vec{i} + x\vec{j}$$

and therefore

$$\vec{F}(t) = \cos t\sin^2 t\vec{i} + \cos t\vec{j}$$

Then

$$\begin{aligned}\oint_C xy^2dx + xdy &= \int_0^{2\pi} \vec{F}(t) \cdot \vec{r}'(t)dt \\ &= \int_0^{2\pi} \left(-\cos t\sin^3 t + \cos^2 t\right)dt \\ &= \left[-\frac{\sin^4 t}{4} + \frac{1}{2}\cos t\sin t + \frac{1}{2}t\right]_0^{2\pi} \\ &= \pi\end{aligned}$$

(The second formula in the table of integrals was used in the above calculation.

3b $\left[15\ \textbf{pts.}\right]$ Evaluate the line integral in 3a, namely

$$\oint_C xy^2dx + xdy$$

using Green's Theorem.

Solution: Green's Theorem is

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_R (Q_x - P_y)dA$$

Here $P = xy^2$ and $Q = x$ so $Q_x - P_y = 1 - 2xy$

Hence

$$\begin{aligned}\iint_R (Q_x - P_y)dA &= \iint_{x^2+y^2\le 1} (1-2xy)dA \\ &= \int_0^{2\pi}\int_0^1 \left(1 - 2r^2\cos\theta\sin\theta\right) r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \left(r - 2r^3\cos\theta\sin\theta\right) dr\,d\theta \\ &= \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{1}{2}r^4\cos\theta\sin\theta\right]_0^1 d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos\theta\sin\theta\right)d\theta \\ &= \left[\frac{\theta}{2} - \frac{1}{4}\sin^2\theta\right]_0^{2\pi} = \pi\end{aligned}$$

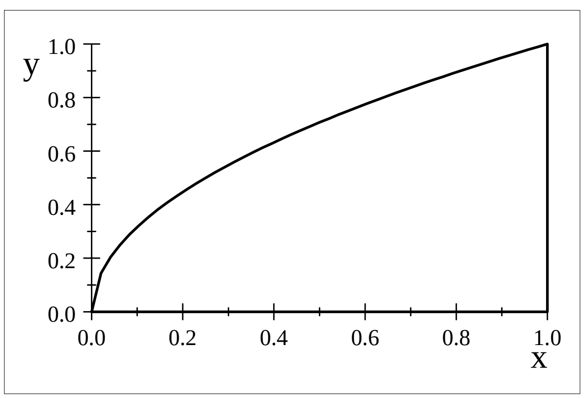
4 [15 **pts**.] Evaluate

$$\oint_C (1+\tan^5 x)dx + \left(x^2 + e^y\right)dy$$

Where C is the positively oriented boundary of the region R enclosed by the curves $y = \sqrt{x}$, $x = 1$, and $y = 0$. Be sure to sketch C.

Solution*:*

The region enclosed by C is shown below.



We use Green's Theorem to evaluate the integral, since C is a closed curve.

$$\begin{aligned}\oint_C (1+\tan^5 x)dx + \left(x^2+e^y\right)dy &= \iint_R \left(\frac{\partial\left(x^2+e^y\right)}{\partial x} - \frac{\partial(1+\tan^5 x)}{\partial y}\right)dA \\ &= \int_0^1\int_0^{\sqrt{x}} (2x-0)\,dy\,dx = 2\int_0^1 x^{\frac{3}{2}}dx = \frac{4}{5}\end{aligned}$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$