Ma 227 **Exam II B Solutions** **11/13/12**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

1 a [20 **pts**.] Evaluate

$$\int_C x^4 y ds$$

where C is the top half of the circle, $x^2 + y^2 = 16$ traversed in the counter clockwise direction.
Solution: We first need a parameterization of the circle. This is given by,

$$x = 4\cos t \quad y = 4\sin t$$

The range of t that gives the right half of the circle traversed in the counter clockwise direction is

$$0 \le t \le \pi$$

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -4\sin t \quad \frac{dy}{dt} = 4\cos t$$

Thus

$$\int_C x^4 y ds = \int_0^\pi \left((4\cos t)^4 (4\sin t)\right)\sqrt{(-4\sin t)^2 + (4\cos t)^2}\, dt$$

$$= 4^6 \int_0^\pi \left(\cos^4 t\right)(\sin t)dt = -4^6 \frac{\cos^5 t}{5}\Big|_0^\pi = \frac{-4^6}{5}(-1-(1)) = \frac{2\left(4^6\right)}{5} = \frac{8192}{5}$$

1b [**15 pts**.] Let

$$\vec{F}(x,y,z) = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$$

Show that

$$\text{div}\left(\text{curl}\vec{F}\right) = \nabla \cdot \left(\nabla \times \vec{F}\right) = 0$$

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{matrix}$$

$$= \frac{\partial F_3}{\partial y}\vec{i} + \frac{\partial F_1}{\partial z}\vec{j} + \frac{\partial F_2}{\partial x}\vec{k} - \frac{\partial F_1}{\partial y}\vec{k} - \frac{\partial F_2}{\partial z}\vec{i} - \frac{\partial F_3}{\partial x}\vec{j}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Therefore

$$\nabla \bullet \left(\nabla \times \vec{F}\right) = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0$$

2 [20 **pts**.] Find a function $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}(x,y,z) = \left(2x\sin y - 2y^3\right)\vec{i} + \left(x^2\cos y - 6xy^2 + 3y^2e^{2z}\right)\vec{j} + \left(5 + 2y^3e^{2z}\right)\vec{k}$$

Solution: Not part of the solution. Note that

$$\nabla \times \left(2x\sin y - 2y^3, x^2\cos y - 6xy^2 + 3y^2e^{2z}, 5 + 2y^3e^{2z}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ so this force field is conservative

and such a $\phi(x,y,z)$ exists.

$$\phi_x = 2x\sin y - 2y^3$$

so integrating with respect to x and holding y and z fixed, we have

$$\phi(x,y,z) = x^2\sin y - 2xy^3 + g(y,z)$$

Differentiating with respect to y we have

$$\phi_y = x^2\cos y - 6xy^2 + g_y = x^2\cos y - 6xy^2 + 3y^2e^{2z}$$

and

$$g_y = 3y^2e^{2z}$$

Integrating with respect to y while holding z fixed we have

$$g = y^3e^{2z} + h(z)$$

Thus

$$\phi(x,y,z) = x^2\sin y - 2xy^3 + y^3e^{2z} + h(z)$$

Differentiating with respect to z we have

$$\phi_z = 2y^3e^{2z} + h'(z) = 5 + 2y^3e^{2z}$$

Therefore $h'(z) = 5$ and $h(z) = 5z + K$, where K is a constant. Hence finally we have

$$\phi(x,y,z) = x^2\sin y - 2xy^3 + y^3e^{2z} + 5z + K$$

Any value of K may be chosen, since only one function is required.

3a [15 **pts**.] Evaluate

$$\oint_C 2yx^2dx - 2xy^2dy$$

directly without using Green's Theorem, where C is the positively oriented circle of radius 2 centered at the origin.

Solution: Let

$$x = 2\cos t, \;\; y = 2\sin t \text{ where } \;\; 0 \le t \le 2\pi$$

Then

$$\begin{aligned}\oint_C 2yx^2dx - 2xy^2dy &= \int_0^{2\pi}\left[2(2\sin t)(2\cos t)^2(-2\sin t) - 2(2\cos t)(2\sin t)^2(2\cos t)\right]dt \\ &= -64\int_0^{2\pi}\left(\sin^2 t\cos^2 t\right)dt \\ &= -64\left(\frac{1}{8}x - \frac{1}{32}\sin 4x\right)_0^{2\pi} \\ &= -16\pi\end{aligned}$$

3b [15 **pts**.] Evaluate the line integral in 3a, namely

$$\oint_C 2yx^2dx - 2xy^2dy$$

using Green's Theorem.

Solution: Here

$$P = 2yx^2 \;\text{ and }\; Q = -2xy^2$$

Thus

$$P_y = 2x^2 \;\text{ and }\; Q_x = -2y^2$$

Hence

$$\begin{aligned}\oint_C 2yx^2dx - 2xy^2dy &= \iint_{x^2+y^2\le 4}\left(-2y^2 - 2x^2\right)dA \\ &= -2\int_0^{2\pi}\int_0^2 r^2 r dr d\theta \\ &= -2\int_0^{2\pi}\frac{r^4}{4}\Big|_0^2 d\theta = -8\int_0^{2\pi}d\theta = -16\pi\end{aligned}$$

4 [15 **pts**.] Use Green's Theorem to find the area of ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Solution:

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

We parametrize the curve using

$$x = 3\cos t \quad y = 2\sin t \quad 0 \le t \le 2\pi$$

Hence

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi}(3\cos t(2\cos t) - 2\sin t(-3\sin t))dt \\
&= \frac{1}{2}\int_0^{2\pi} 6\left(\cos^2 t + \sin^2 t\right)dt = \frac{6}{2}\int_0^{2\pi} dt = 6\pi
\end{aligned}
$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^2 x \cos^2 x dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int \sin^4 t dt = \frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C$
$\int \cos^4 t dt = \frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$