Ma 227 **Exam II B Solutions** **11/5/13**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

1a [20 **pts**.] Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = z\vec{i} + y^2\vec{j} + x\vec{k}$ and C is the curve given by

$$C: \; x(t) = t^2, y(t) = e^t, z(t) = t+1 \quad 0 \le t \le 2$$

Solution:

$$\vec{F}(t) = (t+1)\vec{i} + e^{2t}\vec{j} + t^2\vec{k}$$

$$\vec{r}(t) = t^2\vec{i} + e^t\vec{j} + (t+1)\vec{k}$$

so

$$\vec{r}\,'(t) = 2t\vec{i} + e^t\vec{j} + \vec{k}$$

Thus

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^2 \vec{F}(t) \cdot \vec{r}\,'(t)dt \\ &= \int_0^2 \left(2t^2 + 2t + e^{3t} + t^2\right)dt \\ &= \int_0^2 \left(3t^2 + 2t + e^{3t}\right)dt \\ &= \left[t^3 + t^2 + \frac{1}{3}e^{3t}\right]_0^2 = 8 + 4 + \frac{1}{3}e^6 - \frac{1}{3} = \frac{1}{3}\left(35 + e^6\right)\end{aligned}$$

1b [**15 pts**.] If the function $f(x,y,z)$ has continuous second-order partial derivatives, show that

$$\text{curl}\left(\text{grad } f\right) = 0$$

Solution: $\nabla f = \text{grad } f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ so

$$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{matrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{k} = \vec{0}$$

2 [20 **pts**.] Find a function $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}(x,y,z) = \left(y^2z^{-1}\right)\vec{i} + \left(4yz + 2xyz^{-1}\right)\vec{j} + \left(2y^2 - xy^2z^{-2}\right)\vec{k}$$

Solution:

$$\phi_x = y^2 z^{-1} \quad \phi_y = 4yz + 2xyz^{-1} \quad \phi_z = 2y^2 - xy^2 z^{-2}$$

Starting with ϕ_x and integrating with respect to x we have

$$\phi(x,y,z) = xy^2 z^{-1} + g(y,z)$$

Hence

$$\phi_y = 2xyz^{-1} + g_y = 4yz + 2xyz^{-1}$$

Therefore

$$g_y = 4yz$$
$$g(y,z) = 2y^2 z + h(z)$$

andy

$$\phi(x,y,z) = xy^2 z^{-1} + 2y^2 z + h(z)$$

Therefore

$$\phi_z = -xy^2 z^{-2} + 2y^2 + h'(z) = 2y^2 - xy^2 z^{-2}$$

Hence $h(z) = K$ where K is any constant. Finally we have

$$\phi(x,y,z) = xy^2 z^{-1} + 2y^2 z + K$$

3a [15 **pts**.] Evaluate

$$\oint_C y\,dx + x^2 y\,dy$$

directly without using Green's Theorem, where C is the positively oriented circle of radius 1 centered at the origin.

Solution: C may be parametrized as $x = \cos t, y = \sin t \quad 0 \le t \le 2\pi$ so $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j}$ and $\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j}$. We let

$$\vec{F}(x,y) = y\vec{i} + x^2 y\vec{j}$$

and therefore

$$\vec{F}(t) = \sin t\vec{i} + \cos^2 t \sin t\vec{j}$$

Then

$$\begin{aligned}
\oint_C y\,dx + x^2 y\,dy &= \int_0^{2\pi} \vec{F}(t) \cdot \vec{r}'(t)\,dt \\
&= \int_0^{2\pi} \left(-\sin^2 t + \cos^3 t \sin t\right) dt \\
&= \left[-\left(-\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right) - \frac{1}{4}\cos^4 t \right]_0^{2\pi} \\
&= -\pi
\end{aligned}$$

(The first formula in the table of integrals was used in the above calculation.

3b [15 **pts**.] Evaluate the line integral in 3a, namely

$$\oint_C y\,dx + x^2 y\,dy$$

using Green's Theorem.

Solution: Green's Theorem is

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_R (Q_x - P_y)dA$$

Here $P = y$ and $Q = x^2y$ so $Q_x - P_y = 2xy - 1$

Hence

$$\begin{aligned}\iint_R (Q_x - P_y)dA &= \iint_{x^2+y^2\le 1} (2xy-1)dA \\ &= \int_0^{2\pi}\int_0^1 \left(2r^2\cos\theta\sin\theta - 1\right)rdrd\theta \\ &= \int_0^{2\pi}\int_0^1 \left(2r^3\cos\theta\sin\theta - r\right)drd\theta \\ &= \int_0^{2\pi}\left[\frac{1}{2}r^4\cos\theta\sin\theta - \frac{r^2}{2}\right]_0^1 d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{2}\cos\theta\sin\theta - \frac{1}{2}\right)d\theta \\ &= \left[\frac{1}{4}\sin^2\theta - \frac{\theta}{2}\right]_0^{2\pi} = -\pi\end{aligned}$$

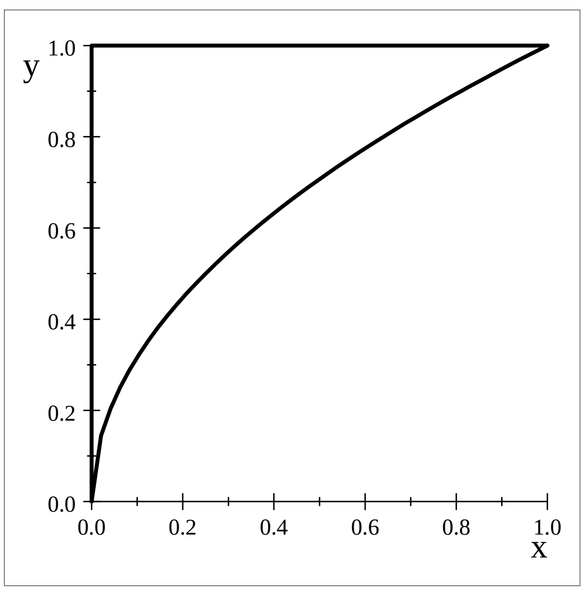
4 [15 **pts**.] Evaluate

$$\oint_C (y + \tan^5 x)dx + \left(2x^2 + e^{2y}\right)dy$$

Where C is the positively oriented boundary of the region R enclosed by the curves $y = \sqrt{x}$, $x = 0$, and $y = 1$. Be sure to sketch C.

Solution:

The region enclosed by C is shown below.



We use Green's Theorem to evaluate the integral, since C is a closed curve.

$$\oint_C (y+\tan^5 x)dx + \left(2x^2+e^{2y}\right)dy = \iint\limits_R \left(\frac{\partial\left(2x^2+e^{2y}\right)}{\partial x} - \frac{\partial(y+\tan^5 x)}{\partial y} \right) dA$$

$$= \int_0^1 \int_0^{y^2} (4x-1)dxdy = \int_0^1 \left[2x^2 - x\right]_{x=0}^{x=y^2} dy$$

$$= \int_0^1 2y^4 - y^2 dy = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}.$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$