Ma 227 **Exam III A Solutions** **12/4/06**

Name: ________________________________

Lecture Section: ________________________________

I pledge my honor that I have abided by the Stevens Honor System. ________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2a ________

#2b ________

#2c ________

#3a ________

#3b ________

#4a ________

#4b ________

Total Score ________

1a [**15 pts**.] Find the work done by the force field

$$\vec{F}(x,y,z) = -\frac{1}{2}x\vec{i} - \frac{1}{2}y\vec{j} + \frac{1}{4}\vec{k}$$

along the plane path $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ from the point $(1,0,0)$ to $(-1,0,3\pi)$.
Solution:

$$\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j} + \vec{k} \quad 0 \le t \le 3\pi$$

$$\vec{F}(t) = -\frac{1}{2}\cos t\vec{i} - \frac{1}{2}\sin t\vec{j} + \frac{1}{4}\vec{k}$$

so

$$\vec{F}(t)\cdot\vec{r}'(t) = \frac{1}{2}\sin t\cos t - \frac{1}{2}\sin t\cos t + \frac{1}{4} = \frac{1}{4}$$

Therefore

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{3\pi}\frac{1}{4}dt = \frac{3}{4}\pi$$

2a [**10 pts**.] Show that $\int_C \vec{F}\cdot d\vec{r}$ is independent of path for the force field

$$\vec{F} = e^x\cos y\vec{i} - e^x\sin y\vec{j} + 2\vec{k}$$

Solution:

$$\operatorname{curl}\vec{F} = \left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\cos y & -e^x\sin y & 2 \end{array}\right|\begin{array}{cc} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ e^x\cos y & -e^x\sin y \end{array}$$

$$= 0\vec{i} + 0\vec{j} - e^x\sin y\vec{k} + e^x\sin y\vec{k} + 0\vec{i} + 0\vec{j} = 0$$

Therefore this force field is conservative and for this $\vec{F}$ the $\int_C \vec{F}\cdot d\vec{r}$ is path independent.

2b [10 **pts**.] Find a function $\Phi(x,y,z)$ such that $\nabla\Phi = \vec{F}$, where $\vec{F}$ is the force field above.
Solution:

$$\Phi_x = e^x\cos y \quad \Phi_y = -e^x\sin y \quad \Phi_z = 2$$

so

$$\Phi = e^x\cos y + g(y,z)$$

Therefore

$$\Phi_y = -e^x\sin y + g_y = -e^x\sin y$$

so $g_y = 0$ and $g(y,z) = h(z)$. Then

$$\Phi = e^x\cos y + h(z)$$

Now

$$\Phi_z = h'(z) = 2$$

so

$$\Phi = e^x\cos y + 2z + k$$

2c[10 **pts**.] Find the work done by the force field given in 2a along a curve C from $\left(0,\frac{\pi}{2},1\right)$ to

$(1,\pi,3)$.

Solution:

$$\text{Work} = \int_c \vec{F} \cdot d\vec{r} = \int_{\left(0,\frac{\pi}{2},1\right)}^{(1,\pi,3)} \vec{F} \cdot d\vec{r} = \Phi(1,\pi,3) - \Phi\left(0,\frac{\pi}{2},1\right) = -e + 6 + k - (0 + 2 + k) = 4 - e$$

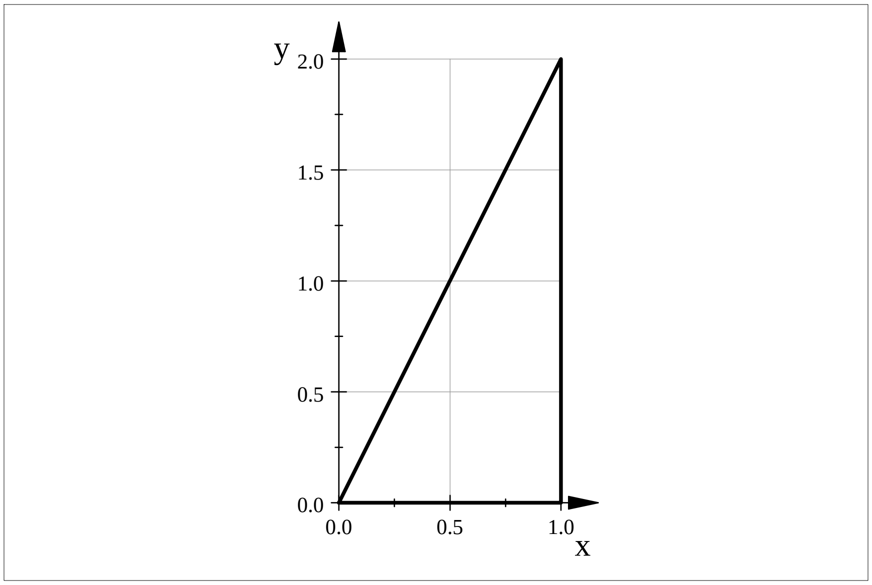
3a [15 **pts**.] Evaluate

$$\oint_C xydx + x^2y^3dy$$

where C is the the triangle with vertices at $(0,0), (1,0)$ and $(1,2)$ traversed counter-clockwise. Sketch C

Solution:

$(0,0,1,0,1,2,0,0)$



Let C_1 denote the segment from $(0,0)$ to $(1,0)$. Then $C_1 : x = t, y = 0$ so $dx = dt$ and $dy = 0$ $0 \le t \le 1$.

Let C_2 denote the segment from $(1,0)$ to $(1,2)$. Then $C_2 : x = 1, y = t$ so $dx = 0$ and $dy = dt$ $0 \le t \le 2$

Let C_3 denote the segment from $(1,2)$ to $(0,0)$. This is the line $y = 2x$. Let $x = t, y = 2t$ $dx = dt, dy = 2dt$ where t goes from 1 to 0.

Then

$$\begin{aligned}\oint_C xydx + x^2y^3dy &= \oint_{C_1+C_2+C_3} xydx + x^2y^3dy \\ &= \int_0^1 0dt + \int_0^2 t^3dt + \int_1^0 \left(2t^2 + 8t^5(2)\right)dt \\ &= \frac{1}{4}t^4\Big]_0^2 + \left(\frac{2t^3}{3} + \frac{16t^6}{6}\right)\Big]_1^0 = 4 - \frac{2}{3} - \frac{8}{3} = \frac{2}{3}\end{aligned}$$

3b [15 **pts**.] Evaluate the line integral in 3a by using Green's Theorem.

Solution: $P = xy, Q = x^2y^3$.

From Green's Theorem

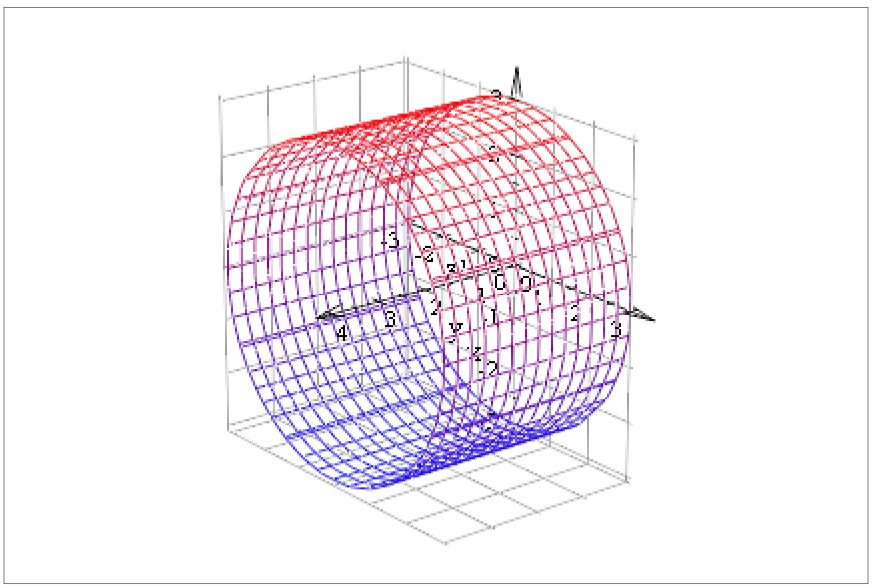
$$\oint P dx + Q dy = \iint_R (Q_x - P_y) dA$$

we have

$$\begin{aligned}\oint_C xy dx + x^2y^3 dy &= \iint_R \left(\frac{\partial\left(x^2y^3\right)}{\partial x} - \frac{\partial(xy)}{\partial y} \right) dA \\ &= \int_0^1 \int_0^{2x} \left(2xy^3 - x\right) dy dx = \int_0^1 \left(\tfrac{1}{2}xy^4 - xy \right) \Big]_0^{2x} dx \\ &= \int_0^1 \left(8x^5 - 2x^2\right) dx = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}\end{aligned}$$

4 a [**10 pts**.] Let S be the portion of the cylinder $y^2 + z^2 = 9$ between $x = 0$ and $x = 4$, $-3 \le z \le 3$. Sketch S and give a parametrization of S.

Solution: $y^2 + z^2 = 9$ is a circle in the y, z –plane, and $-3 \le z \le 3$. We have
$y^2 + z^2 = 9$



Since $y^2 + z^2 = 9$, $-3 \le z \le 3$ is a circle in the first and fourth quadrants of the y, z –plane we use

$$x = x, y = 3\cos t, z = 3\sin t \quad 0 \le x \le 4,\ \ 0 \le t \le 2\pi$$

so for the parametrization we have

$$\vec{r}(x,t) = x\vec{i} + 3\cos t\vec{j} + 3\sin t\vec{k} \ \ 0 \le x \le 4,\ \ 0 \le t \le 2\pi$$

4 b [**15 pts**.] Give an expression for

$$\iint_S (x + z) dS$$

where S is the surface in part 4a. Do *not* evaluate your expression.
Solution:

$$\vec{r}_x = \vec{i}$$

$$\vec{r}_t = -3\sin t\vec{j} + 3\cos t\vec{k}$$

$$\vec{r}_x \times \vec{r}_t = \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & -3\sin t & 3\cos t \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ 1 & 0 \\ 0 & -3\sin t \end{array}$$

$$= -3\sin t\vec{k} - 3\cos t\vec{j}$$

Therefore

$$|\vec{r}_x \times \vec{r}_t| = 3$$

Hence

$$\iint_S (x+z)dS = \int_0^4 \int_0^{2\pi} (x + 3\sin t)(3)dtdx$$