Ma 227 **Exam III Solutions** **12/5/07**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#2 ________

#3a ________

#3b ________

#4a ________

#4b ________

Total Score ________

1 [**15 pts**.] Evaluate the line integral

$$\int_C y^3 dx + x^2 dy$$

where C is the arc of the parabola $x = 1 - y^2$ from $(0,-1)$ to $(0,1)$.
Solution: We use y as the parameter. Then $dx = -2y dy$ so

$$\begin{aligned}\int_C y^3 dx + x^2 dy &= \int_{-1}^{1} \left[y^3(-2y) + \left(1 - y^2\right)^2 \right] dy \\ &= \int_{-1}^{1} \left(-2y^4 + 1 - 2y^2 + y^4\right) dy \\ &= \int_{-1}^{1} \left(-y^4 - 2y^2 + 1\right) dy \\ &= -\frac{y^5}{5} - \frac{2}{3}y^3 + y \Bigg]_{-1}^{1} \\ &= -\frac{1}{5} - \frac{2}{3} + 1 - \frac{1}{5} - \frac{2}{3} + 1 = -\frac{2}{5} - \frac{4}{3} + 2 = \frac{4}{15}\end{aligned}$$

1b [**15 pts**.] Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Show that

$$\nabla(\ln r) = \frac{\vec{r}}{r^2}$$

Solution: $r = \sqrt{x^2 + y^2 + z^2}$

$$\begin{aligned}\nabla \ln r &= \frac{\partial \ln r}{\partial x}\vec{i} + \frac{\partial \ln r}{\partial y}\vec{j} + \frac{\partial \ln r}{\partial z}\vec{k} \\ &= \frac{1}{2}\left(\frac{2x}{r\sqrt{x^2 + y^2 + z^2}}\vec{i} + \frac{2y}{r\sqrt{x^2 + y^2 + z^2}}\vec{j} + \frac{2z}{r\sqrt{x^2 + y^2 + z^2}}\vec{k} \right) \\ &= \frac{1}{r^2}\vec{r}\end{aligned}$$

2a [**20 pts**.] The vector function

$$\vec{F} = e^y \vec{i} + (xe^y + e^z)\vec{j} + ye^z \vec{k}$$

is conservative. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the line segment from $(0,2,0)$ to $(4,0,3)$.
Solution: Since we are given that the vector function is conservative, then there exists $\Phi(x,y,z)$ such that $\nabla\Phi = \vec{F}$. We find Φ and use it to evaluate the line integral. We have that

$$\Phi_x = e^y \quad \Phi_y = xe^y + e^z \quad \Phi_z = ye^z$$

Integrating Φ_x with respect to x while holding y and z constant yields

$$\Phi = xe^y + g(y,z)$$

Then

$$\Phi_y = xe^y + \frac{\partial g}{\partial y} = xe^y + e^z$$

so

$$\frac{\partial g}{\partial y} = e^z \Rightarrow g(y,z) = ye^z + h(z)$$

Thus

$$\Phi = xe^y + ye^z + h(z)$$

Therefore

$$\Phi_z = ye^z + h'(z) = ye^z$$

Hence $h(z) = K,$ a constant, and

$$\Phi = xe^y + ye^z + K$$

Therefore

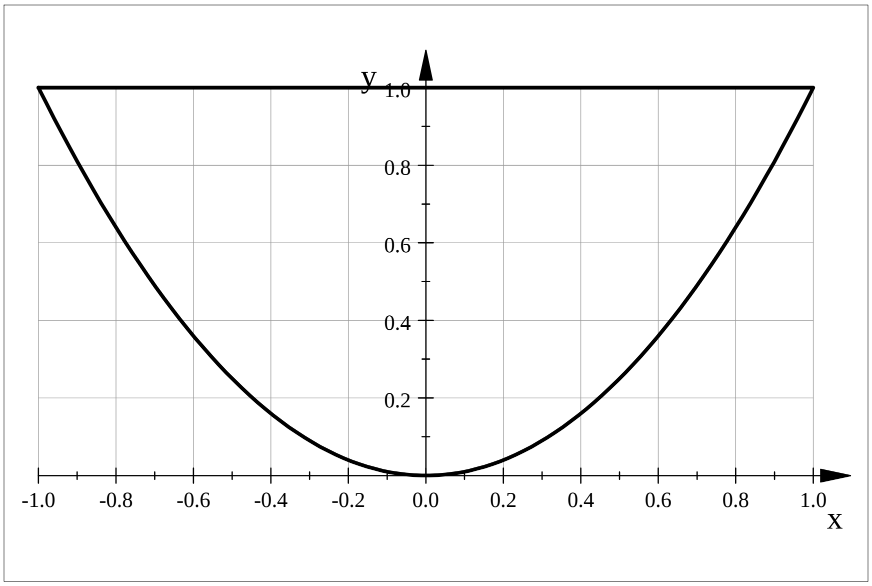
$$\int_C \vec{F} \cdot d\vec{r} = \Phi(4,0,3) - \Phi(0,2,0) = 4 - 2 = 2$$

3a [15 **pts**.] Evaluate

$$\oint_C xy^2dx - x^2ydy$$

directly without using Green's Theorem, where C consists of the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$ and the line segment from $(1,1)$ to $(-1,1)$ Sketch C

Solution: x^2



Let C_1 denote the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$ and C_2 the line joining $(1,1)$ to $(-1,1)$. Let $x = t,$ so $y = t^2$ and $dy = 2tdt$ on C_1 and $x = t$ and $y = 1$ on C_2. Then

$$C_1 : \vec{r}(t) = t\vec{i} + t^2\vec{j}, \quad -1 \le t \le 1 \Rightarrow \vec{r}'(t) = \vec{i} + 2t\vec{j}$$

$$C_2 : \vec{r}(t) = t\vec{i} + \vec{j}, \quad t : 1 \to -1 \Rightarrow \vec{r}'(t) = \vec{i}$$

and $C = C_1 + C_2$.

Let

$$\vec{F}(x,y) = xy^2\vec{i} - x^2y\vec{j}$$

Therefore

$$\oint_C xy^2dx - x^2ydy = \oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F}(t) \cdot \vec{r}'(t)dt + \int_{C_2} \vec{F}(t) \cdot \vec{r}'(t)dt$$

$$= \int_{-1}^{1} \left(t\left(t^2\right) - t^2\left(t^2\right)(2t)\right)dt + \int_{1}^{-1} tdt = 0$$

3b [15 **pts.**] Evaluate the line integral in 3a by using Green's Theorem.

Solution:

$$\oint P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

so

$$\oint_C xy^2dx - x^2ydy = \iint_R \left(\frac{\partial\left(-x^2y\right)}{\partial x} - \frac{\partial\left(xy^2\right)}{\partial y}\right)dA$$

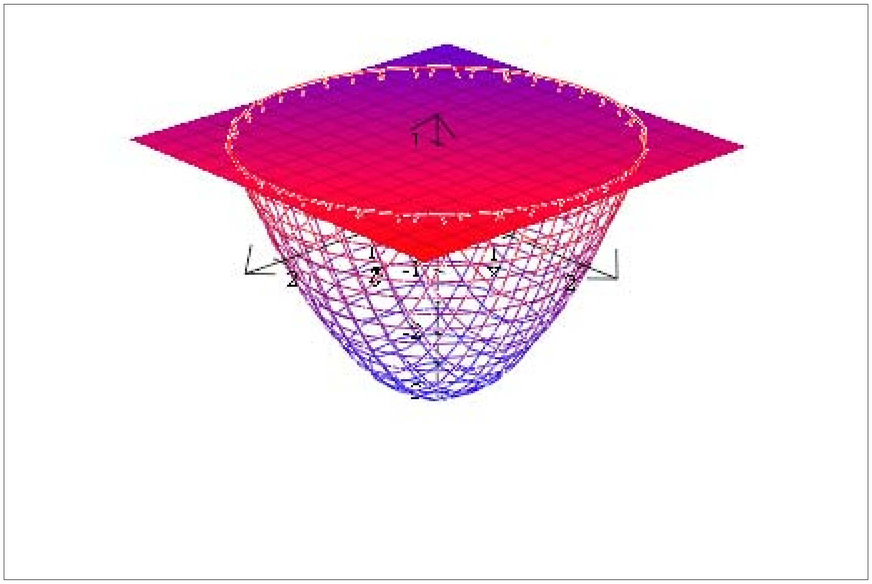
$$= \iint_R (-2xy - 2xy)dA = -4\iint_R xydA$$

$$= -4\int_{-1}^{1}\int_{x^2}^{1} xydydx = -4\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} xydxdy = 0$$

4 a [**5 pts**.] Let S be the part of the paraboloid $z = x^2 + y^2 - 3$ below the plane $z = 1$. Sketch S and give a parametrization of S in rectangular coordinates.

Solution:

$z = x^2 + y^2 - 3$



$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + \left(x^2 + y^2 - 3\right)\vec{k}$$

4 b [**15 pts**.] Give an expression for

$$\iint_S (y + z)dS$$

in rectangular coordinates where S is the surface in part 4a. Do *not* evaluate your expression.
Solution:

$$\vec{r}_x = \vec{i} + 2x\vec{k} \text{ and } \vec{r}_y = \vec{j} + 2y\vec{k}$$

Therefore

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \vec{k} - 2x\vec{i} - 2y\vec{j}$$

Thus $|\vec{r}_x \times \vec{r}_y| = \sqrt{4x^2 + 4y^2 + 1}$ and $z = 1$ implies $x^2 + y^2 = 4$

$$\iint_S (y+z)dS = \iint_{x^2+y^2\leq 4} \left(y + x^2 + y^2 - 3\right)\sqrt{4x^2 + 4y^2 + 1}\, dA$$

Or we can use the formula

$$\iint_S f(x,y,z)dS = \iint_R f(x,y,\phi(x,y))\sqrt{1 + (\phi_x)^2 + (\phi_y)^2}\, dA$$

where R is the projection of the surface $z = \phi(x,y)$ onto the x,y –plane. The surface $z = \phi(x,y) = x^2 + y^2 - 3$ $0 \leq z \leq 1$ projects onto the circle $x^2 + y^2 \leq 1$ in the x,y –plane and $\sqrt{1 + (\phi_x)^2 + (\phi_y)^2} = \sqrt{1 + (2x)^2 + (2y)^2}$. Thus

$$\iint_S (y+z)dS \quad \iint_{x^2+y^2\leq 4} \left(y + x^2 + y^2 - 3\right)\sqrt{1 + (2x)^2 + (2y)^2}\, dA$$