Ma 227 **Exam III Solutions** **12/7/09**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#2 ________

#3a ________

#3b ________

#4a ________

#4b ________

Total Score ________

1a [15 pts.] Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = z\vec{i} + y^2\vec{j} + x\vec{k}$ and C is the curve given by $\vec{r}(t) = (t+1)\vec{i} + e^t\vec{j} + t^2\vec{k},\ 0 \le t \le 2.$
Solution:

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_a^b \vec{F}(t) \cdot \vec{r}\,'(t)dt \\ &= \int_0^2 \left(t^2\vec{i} + e^{2t}\vec{j} + (t+1)\vec{k}\right) \cdot \left(\vec{i} + e^t j + 2t\vec{k}\right)dt \\ &= \int_0^2 \left(t^2 + e^{3t} + 2t^2 + 2t\right)dt \\ &= \int_0^2 \left(3t^2 + e^{3t} + 2t\right)dt = t^3 + \frac{e^{3t}}{3} + t^2\Big|_0^2 = 12 + \frac{e^6}{3} - \frac{1}{3} = \frac{35}{3} + \frac{e^6}{3}\end{aligned}$$

1b [15 pts.] Let $f = f(x,y,z)$ Show that

$$\nabla \times \nabla f = 0$$

Assume that the cross partials of f are equal.
Solution:

$$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$$

$$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{matrix}$$

$$= f_{yz}\vec{i} + f_{zx}\vec{j} + f_{xy}\vec{k} - f_{yx}\vec{k} - f_{zy}\vec{i} - f_{xz}\vec{j} = 0$$

2 [20 pts.] The vector function

$$\vec{F} = (2xy + z)\,\vec{i} + x^2\,\vec{j} + x\,\vec{k}$$

is conservative. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is any path from $(1,-1,2)$ to $(2,2,2)$.
Solution: Since $\vec{F}$ is conservative, then there exists a $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}$. Once we find ϕ, then

$$\int_C \vec{F} \cdot d\vec{r} = \phi(2,2,2) - \phi(1,-1,2)$$

We have that

$$\phi_x = 2xy + z,\ \ \phi_y = x^2, \text{ and } \phi_z = x$$

Integrating the first equation with respect to x while holding y and z constant yields

$$\phi = x^2y + xz + g(y,z)$$

Hence

$$\phi_y = x^2 + g_y = x^2$$

So $g_y = 0$ and $g = h(z)$.

$$\phi = x^2y + xz + h(z)$$

$$\phi_z = x + h'(z) = x$$

Hence $h'(z) = 0$ and $h(z) = c,$ a constant and

$$\phi = x^2y + xz + c$$

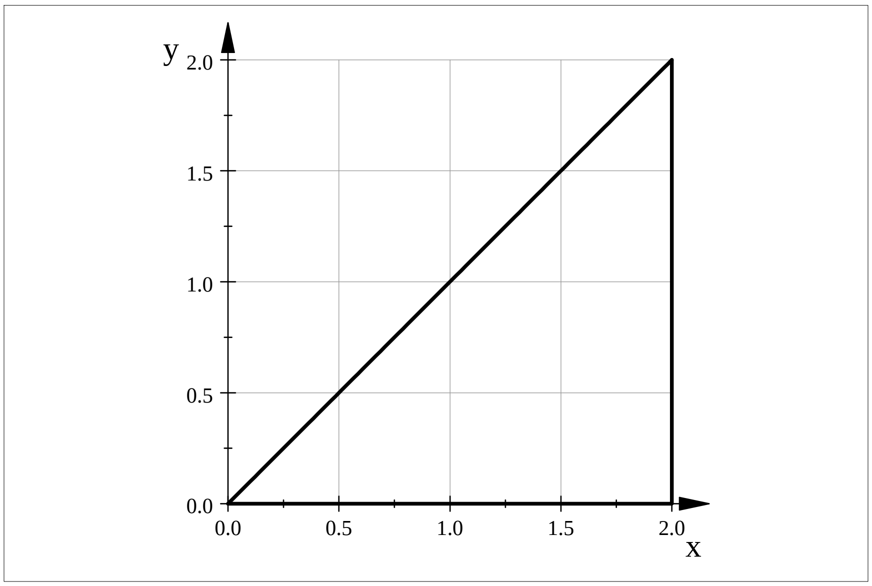
$$\int_C \vec{F} \cdot d\vec{r} = \phi(2,2,2) - \phi(1,-1,2) = 8 + 4 + 1 - 2 = 11$$

Alternatively, you can choose path such as a straight line connecting the points and evaluate the integral.

3a [15 **pts**.] Evaluate

$$\oint_C \sin x dx + x^2y^3 dy$$

directly without using Green's Theorem, where C is the triangle shown below, positively oriented.



Solution: The triangle consists of the segment $y = 0, 0 \le x \le 2,$ followed by the segment $x = 2, 0 \le y \le 2,$ followed by the line $y = x,$ where x goes from 2 to 0. Thus

$$\oint_C \sin x dx + x^2y^3 dy = \int_0^2 \sin x dx + \int_0^2 4y^3 dy + \int_2^0 (\sin x dx + x^5 dx)$$

$$= -\cos 2 + 1 + 2^4 - 1 + \cos 2 - \frac{2^6}{6} = 2^4 - \frac{2^5}{3} = \left(1 - \frac{2}{3}\right)2^4 = \frac{16}{3}$$

3b [15 **pts**.] Evaluate the line integral in 3a by using Green's Theorem.

Solution:

$$\oint_C P dx + Q dy = \iint_R (Q_x - P_y) dA$$

Here $Q = x^2y^3$ and $P = \sin x.$ Therefore

$$\oint_C \sin x dx + x^2 y^3 dy = \iint_R \left(2xy^3 - 0\right) dA$$

$$= \int_0^2 \int_y^2 2xy^3 dxdy = \int_0^2 \int_0^x 2xy^3 dydx = \frac{16}{3}$$

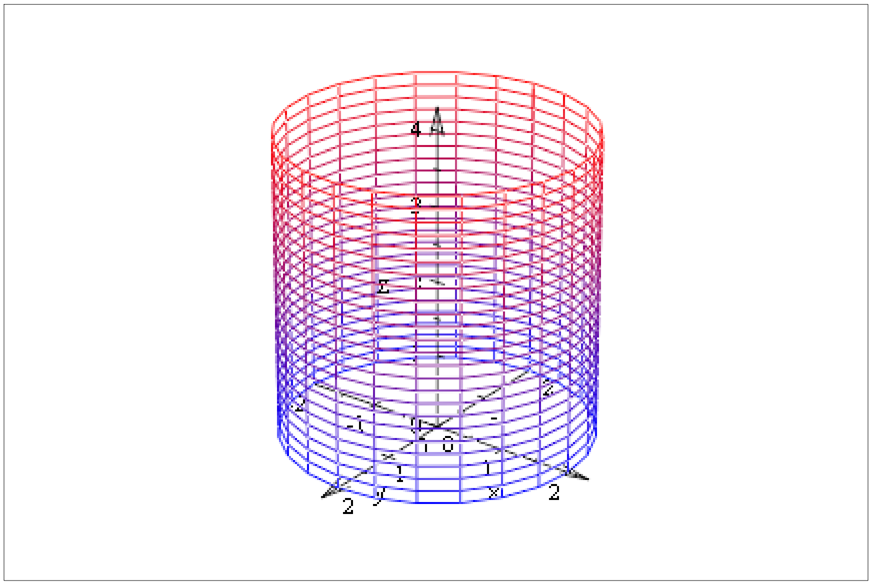
4 a [**10 pts.**] Let S be the surface $x^2 + y^2 = 4, \ 0 \leq z \leq 1$. Sketch S and use cylindrical coordinate to obtain a parametrization the surface. Be sure

to include the ranges of the parameters in the parameter values.

Solution: S is the shell of the cylinder of radius 2 parallel to the z axis between $z = 0$ and $z = 1$. We parametrize S as

$$\vec{r}(\theta, z) = 2\cos\theta\vec{i} + 2\sin\theta\vec{j} + z\vec{k} \ \ 0 \leq \theta \leq 2\pi, \ \ 0 \leq z \leq 1$$

2



4 b [**10 pts.**] Give an expression for

$$\iint_S (x + y + z) dS$$

in cylindrical coordinates where S is the surface in part 4a. Do *not* evaluate your expression.

Solution:

$$\vec{r}_\theta = -2\sin\theta\vec{i} + 2\cos\theta\vec{j}$$
$$\vec{r}_z = \vec{k}$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -2\sin\theta & 2\cos\theta & 0 & -2\sin\theta & 2\cos\theta \\ 0 & 0 & 1 & 0 & 0 \end{array} \right.$$

$$= 2\cos\theta\vec{i} + 2\sin\theta\vec{j}$$

$$|\vec{r}_\theta \times \vec{r}_z| = 2$$

$$\int\int_S f(x,y,z)ds = \int\int_G f(u,v)|\vec{r}_u \times \vec{r}_v|dudv,$$

Thus

$$\iint_S (x+y+z)dS = \int_0^{2\pi}\int_0^1 \left(2\cos\theta + 2\sin\theta + z\right)(2)dzd\theta$$