Ma 227 **Exam III Solutions** **12/6/10**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#2 ________

#3a ________

#3b ________

#4a ________

#4b ________

Total Score ________

1a [**15 pts**.] Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = \sqrt{xy}\,\vec{i} + e^y\vec{j} + xz\vec{k}$ and C is the curve given by $\vec{r}(t) = t^4\vec{i} + t^2\vec{j} + t^3\vec{k},\ 0 \le t \le 1.$

Solution:

$$\vec{F}(t) = \sqrt{t^4\left(t^2\right)}\,\vec{i} + e^{t^2}\vec{j} + t^4\left(t^3\right)\vec{k} = t^3\vec{i} + e^{t^2}\vec{j} + t^7\vec{k}$$

$$\vec{r}'(t) = 4t^3\vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

Hence

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t) \cdot \vec{r}'(t)dt$$

$$= \int_0^1 \left(4t^6 + 2te^{t^2} + 3t^9\right)dt$$

$$= \left[\frac{4}{7}t^7 + e^{t^2} + \frac{3}{10}t^{10}\right]_0^1 = \frac{40}{70} + \frac{21}{10} + e - 1 = e - \frac{9}{70}$$

1b [**15 pts**.] Let $\vec{A}(x,y,z) = 2z\vec{i} + e^x\vec{j} - e^{-y}\vec{k}.$ Calculate $\nabla \cdot \left(\nabla \times \vec{A}\right)$ directly.

Solution:

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & e^x & e^{-y} \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & e^x & e^{-y} \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2z & e^x \end{matrix}$$

$$= -e^{-y}\vec{i} + 2\vec{j} + e^x\vec{k} - 0\vec{k} - 0\vec{i} - 0\vec{j} = -e^{-y}\vec{i} + 2\vec{j} + e^x\vec{k}$$

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = \frac{\partial}{\partial x}(e^{-y}) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(e^x) = 0$$

2 a [**12 pts**.] Find a function $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}(x,y,z) = (2x - y - z)\vec{i} + (2y - x)\vec{j} + (2z - x)\vec{k}$$

Solution:

$$\phi_x = 2x - y - z \qquad \phi_y = 2y - x \qquad \phi_z = 2z - x$$

so integrating ϕ_x with respect to x gives

$$\phi = x^2 - xy - xz + g(y,z)$$

Then

$$\phi_y = -x + g_y = 2y - x$$

and $g(y,z) = y^2 + h(z).$ Hence

$$\phi = x^2 - xy - xz + y^2 + h(z)$$

and

$$\phi_z = -x + h'(z) = 2z - x$$

Thus $h(z) = z^2 + K$ and

$$\phi(x,y,z) = x^2 - xy - xz + y^2 + z^2 + K$$

2 b [**8 pts**.] Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}$ is the vector field in part 2a) and C is the curve given by the vector equation

$$\vec{r}(t) = \left(1 + 2t^2\right)\vec{i} + (1 - t^5)\vec{j} + \left(1 + 2t^6\right)\vec{k} \qquad 0 \le t \le 1$$

Solution: Since $\vec{F}$ is derivable from a potential function, then its line integral is path independent. The curve C begins at $(1.1.1)$ and ends at $(3,0,3)$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = \phi(3,0,3) - \phi(1,1,1) = 8$$

3a $\left[\right.$15 **pts**.$\left.\right]$ Evaluate

$$\oint_C ydx + \left(x + y^2\right)dy$$

directly without using Green's Theorem, where C is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation. (Hint: A parametrization of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x = a\cos t, y = b\sin t$.)

Solution: We may write the given ellipse as $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Hence a parametrization of the ellipse with the given counterclockwise orientation is $x = 3\cos t, y = 2\sin t \;\; 0 \le t \le 2\pi$. Thus

$$\begin{aligned}
\oint_C ydx + \left(x + y^2\right)dy &= \int_0^{2\pi} \left[(2\sin t)(-3\sin t) + \left(3\cos t + 4\sin^2 t\right)(2\cos t)\right]dt \\
&= \int_0^{2\pi} \left(-6\sin^2 t + 6\cos^2 t + 8\sin^2 t\cos t\right)dt \\
&= \int_0^{2\pi} \left[6\left(\cos^2 t - \sin^2 t\right) + 8\sin^2 t\cos t\right]dt \\
&= 3[\cos t\sin t + t + \cos t\sin t - t]_0^{2\pi} + \frac{8}{3}\left[\sin^3 t\right]_0^{2\pi} \text{ from the table} \\
&= 0
\end{aligned}$$

3b $\left[\right.$10 **pts**.$\left.\right]$ Evaluate the line integral in 3a, namely

$$\oint_C ydx + \left(x + y^2\right)dy$$

using Green's Theorem.
Solution: Green's Theorem

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA$$

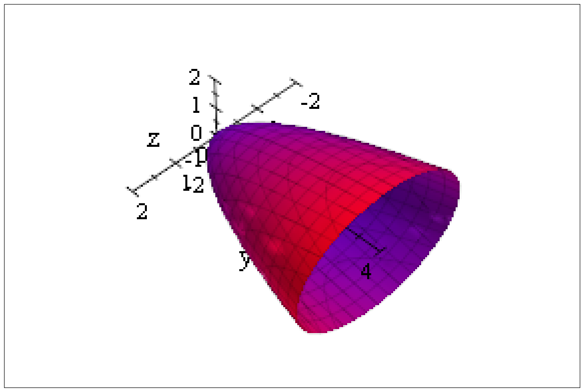
implies

$$\oint_C ydx + \left(x + y^2\right)dy = \iint_{4x^2+9y^2 \le 36} (1 - 1)dA = 0$$

4*a* [10 **pts**] Parametrize the surface S that is the part of the paraboloid

$$y = x^2 + z^2$$

that lies between the planes $y = 4$ and $y = 0$. Sketch the surface S.
Solution:



Let

$$x = u\cos v,\ z = u\sin v,\ \ y = u^2$$

where $0 \le v \le 2\pi$, and $0 \le y \le 4$ implies $0 \le u \le 2$.
Alternative Solution: Use x and z as the parameters, so

$$x = x, y = x^2 + z^2, z = z$$

$0 \le x^2 + z^2 \le 4$.

4*b* [15 **pts**.] Give an iterated double integral equal to

$$\iint_S xz dS$$

where S is the surface in part 3a. Do *not* evaluate your expression.
Solution:

$$\vec{r}(u,v) = u\cos v\vec{i} + u^2\vec{j} + u\cos v\vec{k}$$

so

$$\begin{aligned}\vec{r}_u &= \cos v\vec{i} + 2u\vec{j} + \sin v\vec{k}\\ \vec{r}_v &= -u\sin v\vec{i} + 0\vec{j} + u\cos v\vec{k}\end{aligned}$$

$$\begin{aligned}\vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & 2u & \sin v \\ -u\sin v & 0 & u\cos v \end{vmatrix}\\ &= 2\vec{i}u^2\cos v - \left(\cos^2 v\right)\vec{j}u - u\left(\sin^2 v\right)\vec{j} + 2u^2(\sin v)\vec{k}\\ &= 2u^2\cos v\vec{i} - u\vec{j} + 2u^2\sin v\vec{k}\end{aligned}$$

Thus

$$\begin{aligned}|\vec{r}_u \times \vec{r}_v| &= \sqrt{4u^4\cos^2 v + 4u^4\sin^2 v + u^2}\\ &= u\sqrt{4u^2 + 1}\end{aligned}$$

Thus

$$\iint_S xzdS = \iint_{0\le x^2+z^2\le 4} xz|\vec{r}_u \times \vec{r}_v|dudv$$
$$= \int_0^{2\pi}\int_0^2 (u\cos v)(u\cos v)\left(u\sqrt{4u^2+1}\right)dudv$$

Alternative Solution:

$$\iint_S xzdS = \iint_{0\le x^2+z^2\le 4} xz\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA_{xz}$$
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} xz\sqrt{1+4x^2+4z^2}\,dzdx$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$