Ma 227 **Exam III A Solutions** **4/25/05**

Name: ______________________________

Lecture Section: ______________________________ **Lecturer**: ________________________

I pledge my honor that I have abided by the Stevens Honor System. ________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

Total Score ________

1a [15 pts.] Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, if

$$\vec{F}(x,y) = xz\vec{i} - yz\vec{k}$$

where C is the plane path $x(t) = 4t - 1, y(t) = 2 - 2t, z(t) = t,\ 0 \le t \le 1.$
Solution:

$$\vec{r}(t) = (4t-1)\vec{i} + (2-2t)\vec{j} + t\vec{k} \quad 0 \le t \le 1$$

so

$$\vec{r}'(t) = 4\vec{i} - 2\vec{j} + \vec{k}$$

$$\vec{F}(t) = (4t-1)t\vec{i} - (2-2t)t\vec{k}$$

Thus

$$\vec{F} \cdot \vec{r}'(t) = 16t^2 - 4t - 2t + 2t^2 = 18t^2 - 6t$$

Hence

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(18t^2 - 6t\right) dt = \left[6t^3 - 3t^2\right]_0^1 = 6 - 3 = 3$$

1b [15 pts.] Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Show that

$$\nabla(\ln r) = \frac{\vec{r}}{r^2}$$

Solution: $r = \sqrt{x^2 + y^2 + z^2}$

$$\begin{aligned}\nabla \ln r &= \frac{\partial \ln r}{\partial x}\vec{i} + \frac{\partial \ln r}{\partial y}\vec{j} + \frac{\partial \ln r}{\partial z}\vec{k} \\ &= \frac{1}{2}\left(\frac{2x}{r\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{2y}{r\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{2z}{r\sqrt{x^2+y^2+z^2}}\vec{k}\right) \\ &= \frac{1}{r^2}\vec{r}\end{aligned}$$

2a $\left[20\ \textbf{pts.}\right]$ Find a function $\Phi(x,y,z)$ such that $\nabla\Phi = \vec{F}$, where

$$\vec{F}(x,y,z) = y^2z^3\vec{i} + \left(2xyz^3 + y^2\right)\vec{j} + 3xy^2z^2\vec{k}$$

Solution:

$$\Phi_x = y^2z^3$$

$\Rightarrow$

$$\Phi = xy^2z^3 + g(y,z)$$

$\Rightarrow$

$$\Phi_y = 2xyz^3 + \frac{\partial g}{\partial y} = 2xyz^3 + y^2$$

$\Rightarrow$

$$\frac{\partial g}{\partial y} = y^2 \Rightarrow g(y,z) = \frac{y^3}{3} + h(z)$$

Thus

$$\Phi = xy^2z^3 + \frac{y^3}{3} + h(z)$$

and

$$\Phi_z = 3xy^2z^2 + h'(z) = 3xy^2z^2$$

so $h'(z) = 0 \Rightarrow h(z) = K$ and

$$\Phi = xy^2z^3 + \frac{y^3}{3} + K$$

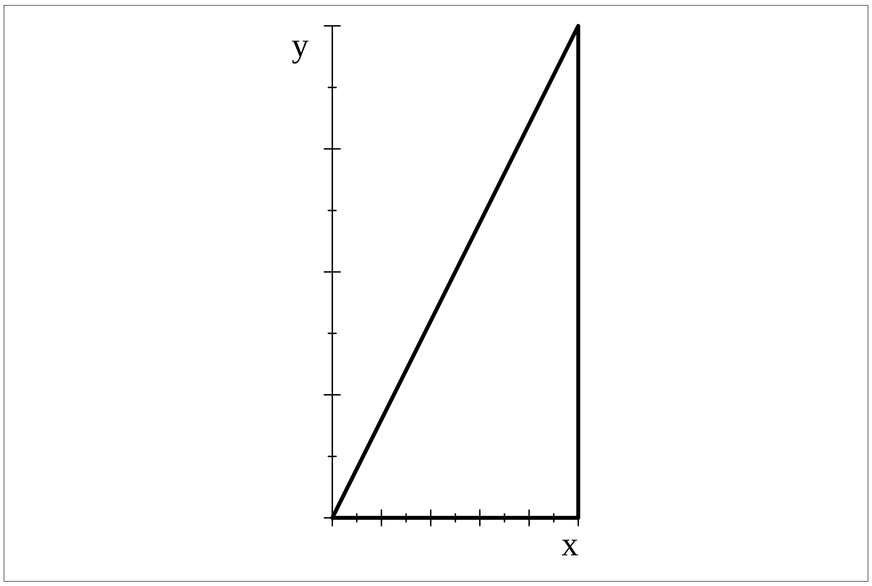
2b [**20 pts**.] Verify that Green's Theorem is true for the line integral

$$\oint_C xydx + x^2y^3dy$$

where C is the triangle with vertices $(0,0),(1,0),(1,2)$.

Solution: The curve C and the region of integration are shown below

$(0,0,1,0,1,2,0,0)$



The line joining $(0,0)$ to $(1,2)$ is $y = 2x$. Now we must show

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA$$

$P = xy$ and $Q = x^2y^3$. Let C_1 be the segment joining $(0,0)$ and $(1,0)$, C_2 the segment joining $(1,0)$ and $(1,2)$ and C_3 the segment joining $(1,2)$ and $(0,0)$.

$$\oint_C xydx + x^2y^3dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = \int_0^1 0dx + \int_0^2 (1)^2y^3dy + \int_1^0 \left[x(2x)dx + x^2(2x)^3(2dx)\right]$$

$$= \left[\frac{y^4}{4}\right]_0^2 + \left[\frac{2}{3}x^3 + 16\frac{x^6}{6}\right]_1^0 = 4 - \frac{2}{3} - \frac{8}{3} = \frac{2}{3}$$

Also

$$\iint_R (Q_x - P_y)\,dA = \iint_R \left(2xy^3 - x\right)dA = \int_0^1 \int_0^{2x} \left(2xy^3 - x\right)dy\,dx$$

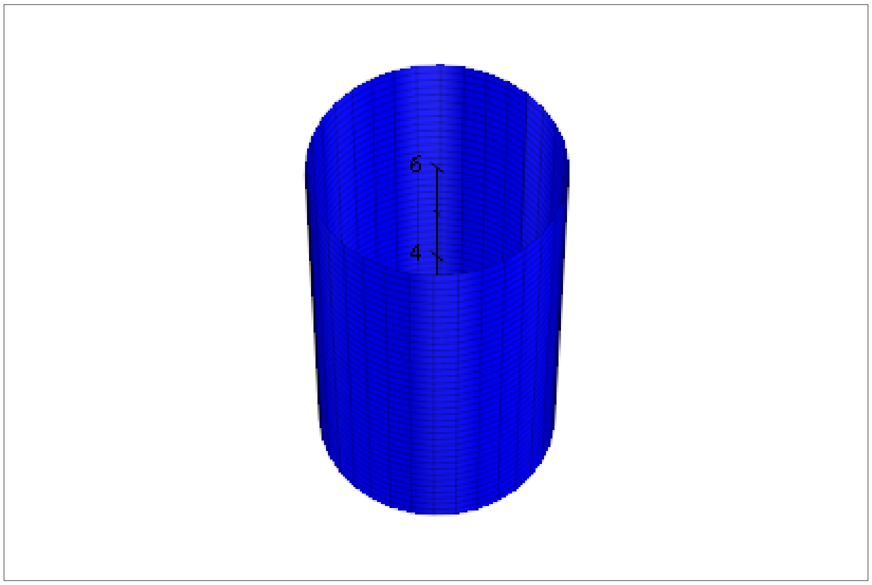
$$= \int_0^1 \left[\frac{xy^4}{2} - xy \right]_0^{2x} dx = \int_0^1 \left[8x^5 - 2x^2 \right] dx = \left[\frac{4x^6}{3} - \frac{2x^3}{3} \right]_0^1 = \frac{2}{3}$$

3 a [**10 pts**.] Let S be the portion of the cylinder $x^2 + y^2 = 3$ that lies between $z = 0$ and $z = 6$. Use cylindrical coordinates to

to give a parametrization of S. Sketch the surface S.
Solution: Let $x = \sqrt{3}\cos\theta, y = \sqrt{3}\sin\theta, z = z,\ \ 0 \le z \le 6,\ \ 0 \le \theta \le 2\pi$ or

$$\vec{r}(\theta,z) = \sqrt{3}\cos\theta\vec{i} + \sqrt{3}\sin\theta\vec{j} + z\vec{k}\quad 0 \le z \le 6,\ 0 \le \theta \le 2\pi$$

$\left(\sqrt{3},\theta,z\right)$



3 b [**20 pts**.] Give an expression for

$$\iint_S y dS$$

where S is the surface in part 3a. Do *not* evaluate your expression.
Solution: For a surface given by

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

that

$$\int\int_S f(x,y,z)ds = \int\int_G f(u,v)|\vec{r}_u \times \vec{r}_v|dudv,$$

where G is the image of the surface S in the u,v -plane. Letting $u = \theta$ and $v = z$, we have

$$\vec{r}(\theta,z) = \sqrt{3}\cos\theta\vec{i} + \sqrt{3}\sin\theta\vec{j} + z\vec{k}$$

and

$$\vec{r}_\theta(\theta,z) = -\sqrt{3}\sin\theta\vec{i} + \sqrt{3}\cos\theta\vec{j}$$
$$\vec{r}_z(\theta,z) = \vec{k}$$

Thus

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sqrt{3}\sin\theta & \sqrt{3}\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \sqrt{3}\cos\theta\vec{i} + \sqrt{3}\sin\theta\vec{j}$$

so

$$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{3}$$

Thus

$$\iint_S y\,dS = \int_0^{2\pi}\int_0^6 \sqrt{3}\sin\theta\left(\sqrt{3}\right)dz\,d\theta = 3\int_0^{2\pi}\int_0^6 \sin\theta\,dz\,d\theta$$