Ma 227 **Exam III A Solutions** **4/26/06**

Name: ______________________________ **ID**: ________________________

Lecture Section: ______________________________ **Lecturer**: ________________________

I pledge my honor that I have abided by the Stevens Honor System. ________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 ________

Total Score ________

1a [15 pts.] Find the work done by the force field

$$\vec{F}(x,y) = (x-y)\vec{i} + 2xy\vec{j}$$

along the plane path that is the graph of $y = 2x^3 - 1$ from $A = (0,-1)$ to $B = (1,1)$.
Solution: The path may be parametrized as $C : x = t, y = 2t^3 - 1 \;\; 0 \le t \le 1$. Then
$\vec{r}(t) = t\vec{i} + (2t^3-1)\vec{j},\ \vec{r}'(t) = \vec{i} + 6t^2\vec{j},\ \ \vec{F}(t) = (t - 2t^3 + 1)\vec{i} + 2t(2t^3-1)\vec{j}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t) \cdot \vec{r}'(t)dt = \int_0^1 \left(t - 2t^3 + 1 + 12t^3(2t^3-1)\right)dt = \int_0^1 (24t^6 - 14t^3 + t + 1)dt$$

$$= \frac{24t^7}{7} - \frac{14t^4}{4} + \frac{t^2}{2} + t\Big]_0^1 = \frac{24}{7} - \frac{7}{2} + \frac{1}{2} + 1 = \frac{10}{7}$$

1b [15 pts.] Consider

$$\vec{F} = (3\sin x - e^y)\vec{i} + (4\arctan x - 12y)\vec{j} + (e^{\cos x} + 4\cos(2z))\vec{k}$$

Find

$$\nabla\left(div(\vec{F})\right) = \nabla\left(\nabla \cdot \vec{F}\right).$$

Solution: $\quad div(\vec{F}) = \frac{\partial}{\partial x}(3\sin x - e^y) + \frac{\partial}{\partial y}(4\arctan x - 12y) + \frac{\partial}{\partial z}(e^{\cos x} + 4\cos(2z))$

$$= 3\cos x - 12 - 8\sin(2z)$$

$$\nabla(div(\vec{F})) = -3\sin x\,\vec{i} + 0\vec{j} - 16\cos(2z)\vec{k}$$

2a $\left[20\ \textbf{pts.}\right]$ Find a function $\Phi(x,y,z)$ such that $\nabla\Phi = \vec{F}$, where

$$\vec{F}(x,y,z) = \left(2xyz + e^{2y}\right)\vec{i} + \left(x^2z + 2xe^{2y} + z^2\sin y\right)\vec{j} + \left(x^2y - 2z\cos y + 2\right)\vec{k}$$

Solution: We first note the

$$\nabla \times \left(2xyz + e^{2y}, x^2z + 2xe^{2y} + z^2\sin y, x^2y - 2z\cos y + 2\right) = (0,0,0)$$

so such a Φ exists. This is not required in order to get full credit for the solution.
We have that

$$\Phi_x = 2xyz + e^{2y} \quad \Phi_y = x^2z + 2xe^{2y} + z^2\sin y \quad \Phi_z = x^2y - 2z\cos y + 2$$

Starting with Φ_x and integrating with respect to x we get

$$\Phi = x^2yz + xe^{2y} + h(y,z)$$

Then

$$\Phi_y = x^2z + 2xe^{2y} + \frac{\partial h}{\partial y} = x^2z + 2xe^{2y} + z^2\sin y$$

Thus

$$\frac{\partial h}{\partial y} = z^2\sin y \Rightarrow h(y,z) = -z^2\cos y + g(z)$$

and therefore

$$\Phi = x^2yz + xe^{2y} - z^2\cos y + g(z)$$

Hence

$$\Phi_z = x^2y - 2z\cos y + g'(z) = x^2y - 2z\cos y + 2$$

Then

$$g'(z) = 2 \Rightarrow g(z) = 2z + k$$

Finally we have that

$$\Phi = x^2yz + xe^{2y} - z^2\cos y + 2z + k$$

Check: SNB $\nabla\left(x^2yz + xe^{2y} - z^2\cos y + 2z + k\right) =$
$\left(2xyz + e^{2y}, x^2z + 2xe^{2y} + z^2\sin y, x^2y - 2z\cos y + 2\right)$

2b [**20 pts**.] Verify that Green's Theorem is true for the line integral

$$\oint_C ydx - xdy$$

where C is the circle with center at the origin and radius 3.
Solution: The circle can be parameterized by

$$x = 3\cos t, \quad y = 3\sin t, \quad 0 \le t \le 2\pi$$

$$\oint_C ydx - xdy = \int_0^{2\pi}(3\sin t(-3\sin t) - 3\cos t(3\cos t))dt = \int_0^{2\pi} -9(\sin^2 t + \cos^2 t)dt = \int_0^{2\pi} -9dt = -18\pi$$

Also since $P = y$ and $Q = -x$,

$$\iint_R (Q_x - P_y)dA = \int\int(-1-1)dA = \int\int -2dA = -2(\text{Area of } x^2 + y^2 = 9) = -2\pi(3)^2 = -18\pi$$

3 a [**10 pts**.] Let S be the portion of $r = \theta^2$ that lies between $z = x^2 + y^2$ and $z = 5.$ Use cylindrical coordinates to give a parametrization of S.

Solution: Let $x = \theta^2\cos\theta, y = \theta^2\sin\theta, z = z,\ (\theta^2)^2 \le z \le 5,\ 0 \le \theta \le 2\pi$ or

$$\vec{r}(\theta,z) = \theta^2\cos\theta\vec{i} + \theta^2\sin\theta\vec{j} + z\vec{k} \quad \theta^4 \le z \le 5,\ 0 \le \theta \le 2\pi$$

3 b [**20 pts**.] Give an expression for

$$\iint_S xdS$$

where S is the surface in part 3a. Do *not* evaluate your expression.
Solution: For a surface given by

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

that

$$\int\int_S f(x,y,z)ds = \int\int_G f(u,v)|\vec{r}_u \times \vec{r}_v|dudv,$$

where G is the image of the surface S in the u,v -plane. Letting $u = \theta$ and $v = z,$ we have

$$\vec{r}(\theta,z) = \theta^2\cos\theta\vec{i} + \theta^2\sin\theta\vec{j} + z\vec{k}$$

and

$$\vec{r}_\theta(\theta,z) = (2\theta\cos\theta - \theta^2\sin\theta)\vec{i} + (2\theta\sin\theta + \theta^2\cos\theta)\vec{j}$$
$$\vec{r}_z(\theta,z) = \vec{k}$$

Thus

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\theta\cos\theta - \theta^2\sin\theta & 2\theta\sin\theta + \theta^2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\theta\sin\theta + \theta^2\cos\theta)\vec{i} - (2\theta\cos\theta - \theta^2\sin\theta)\vec{j}$$

so

$$\begin{aligned} |\vec{r}_\theta \times \vec{r}_z| &= \sqrt{(2\theta\sin\theta + \theta^2\cos\theta)^2 + (2\theta\cos\theta - \theta^2\sin\theta)^2} \\ &= \sqrt{4\theta^2\sin^2\theta + 4\theta^3\sin\theta\cos\theta + \theta^4\cos^2\theta + 4\theta^2\cos^2\theta - 4\theta^3\sin\theta\cos\theta + \theta^4\sin^2\theta} \\ &= \sqrt{4\theta^2 + \theta^4} \end{aligned}$$

Thus

$$\iint_S x\,dS = \int_0^{2\pi}\int_{\theta^4}^{5} \theta^2\cos\theta\left(\sqrt{4\theta^2+\theta^4}\right)dz\,d\theta$$