Ma 227 **Exam III A Solutions** **12/6/11**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1a ________

#1b ________

#2a _________

#2b ________

#3a ________

#3b ________

#4a ________

#4b ________

Total Score ________

1a [**15 pts**.] Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x,y,z) = y\vec{i} + x\vec{j} + z^2\vec{k}$ and C is the curve given by $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k},\ 0 \leq t \leq 2\pi$.
Solution:

$$\vec{F}(t) = \sin t\vec{i} + \cos t\vec{j} + t^2\vec{k}$$

$$\vec{r}'(t) = -\sin t\vec{i} + \cos t\vec{j} + \vec{k}$$

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(t) \cdot \vec{r}(t)dt \\ &= \int_0^{2\pi} \left(-\sin^2 t + \cos^2 t + t^2\right)dt \\ &= \left[\frac{1}{2}\cos t \sin t - \frac{1}{2}t + \frac{1}{2}\cos t \sin t + \frac{1}{2}t + \frac{t^3}{3}\right]_0^{2\pi} \\ &= \cos t \sin t + \frac{t^3}{3}\Big|_0^{2\pi} = \frac{8\pi^3}{3}\end{aligned}$$

1b [**15 pts**.] Let $g(x,y,z)$ be a scalar function whose cross partial derivatives are continuous. Calculate $\nabla \times \nabla g$.

Solution:

$$\nabla g = g_x\vec{i} + g_y\vec{j} + g_z\vec{k}$$

$$\begin{aligned}\nabla \times \nabla g &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ g_x & g_y & g_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \\ &= \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ g_x & g_y & g_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ g_x & g_y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{matrix} \\ &= g_{zy}\vec{i} + g_{xz}\vec{j} + g_{yx}\vec{k} - g_{xy}\vec{k} - -g_{yz}\vec{i} - g_{zx}\vec{j} = 0\end{aligned}$$

2 a [14 **pts**.] Find a function $\phi(x,y,z)$ such that

$$\nabla\phi = \vec{F}(x,y,z) = 2xy^3z^4\vec{i} + \left(3x^2y^2z^4 + y\right)\vec{j} + \left(4x^2y^3z^3 + 2z\right)\vec{k}$$

Solution:

$$\phi_x = 2xy^3z^4$$

$\Rightarrow$

$$\phi = x^2y^3z^4 + g(y,z)$$

Then

$$\phi_y = 3x^2y^2z^4 + g_y = 3x^2y^2z^4 + y$$

Thus

$$g(y,z) = \frac{y^2}{2} + h(z)$$

and

$$\phi = x^2y^3z^4 + \frac{y^2}{2} + h(z)$$

Then

$$\phi_z = 4x^2y^3z^3 + h'(z) = 4x^2y^3z^3 + 2z$$

Hence

$$h(z) = z^2 + K$$

and

$$\phi = x^2y^3z^4 + \frac{y^2}{2} + z^2 + K$$

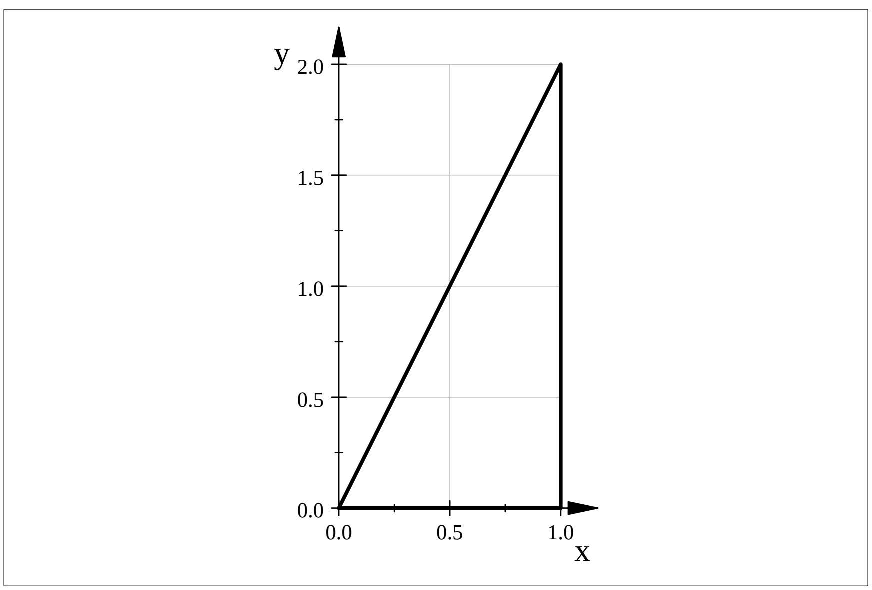
2 b [6 **pts**.] Let Γ be any closed curve in three space. What is the value of $\oint_\Gamma \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is the vector function in 2a. Justify your conclusion.

Solution: Since $\vec{F}$ is derivable from a potential function, then it is a conservative force field, the line integral is independent of path, and $\oint_\Gamma \vec{F} \cdot d\vec{r} = 0$ for any closed curve Γ.

3a $\left[15\ \textbf{pts}.\right]$ Evaluate

$$\oint_C xydx + x^2y^3dy$$

directly without using Green's Theorem, where C is the triangle with vertices $(0,0),(1,0),(1,2)$.
Solution: The triangle C is shown below.
$(0,0,1,0,1,0,1,2,0,0)$



The hypotenuse of the triangle is the line $y = 2x$. We traverse the triangle in the counterclockwise direction.

$$C_1 : x = t,\ \ y = 0\ \ 0 \le t \le 1$$
$$C_2 : x = 1,\ y = t\ \ 0 \le t \le 2$$
$$C_3 : x = t,\ y = 2t\ \ t : 1 \to 0$$

$$\oint_C xydx + x^2y^3dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$$
$$= 0 + \int_0^2 (1)^2 t^3 dt + \int_1^0 t(2t)dt + \int_1^0 t^2(2t)^3(2)dt$$
$$= \frac{t^4}{4}\Big|_0^2 + \frac{2t^3}{3}\Big|_1^0 + 16\frac{t^6}{6}\Big|_1^0$$
$$= 4 - \frac{2}{3} - \frac{8}{3} = \frac{2}{3}$$

3b [10 **pts**.] Evaluate the line integral in 3a, namely

$$\oint_C xydx + x^2y^3dy$$

using Green's Theorem.

Solution: $P = xy$ and $Q = x^2y^3$. Green's Theorem is

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

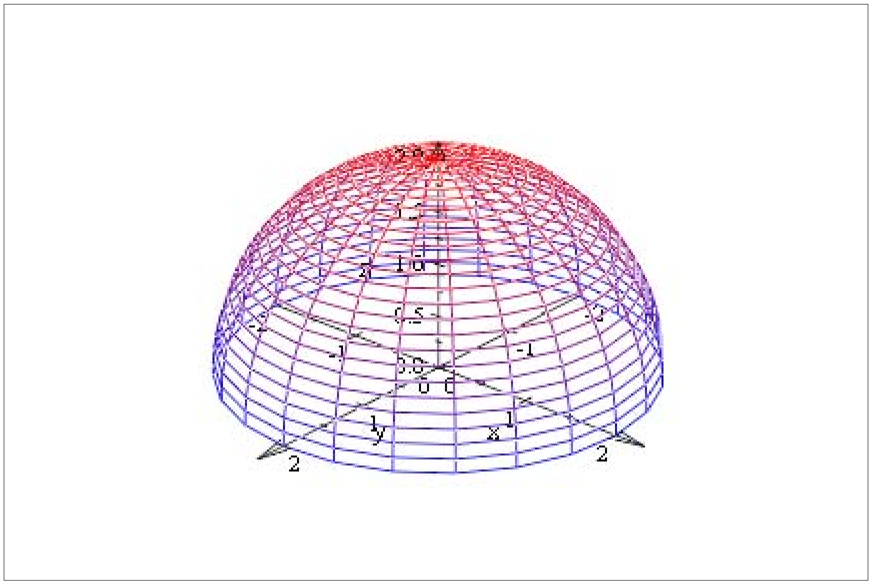
Thus

$$\oint_C xydx + x^2y^3dy = \iint_{\text{Triangle}} \left(2xy^3 - x\right) dA$$
$$= \int_0^1 \int_0^{2x} \left(2xy^3 - x\right) dydx$$
$$= \int_0^1 \left[\frac{xy^4}{2} - xy \right]_0^{2x} dx$$
$$= \int_0^1 \left(8x^5 - 2x^2\right) dx = \frac{4x^6}{3} - \frac{2x^3}{3}\Big|_0^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

4*a* [10 **pts**] Let S be the upper half of the sphere of radius 2 centered at the origin. Sketch and parametrize the surface S. Be sure to include the range of the parameters.

Solution.

$(2, \theta, \phi)$



Since we are dealing with a hemisphere, we use spherical coordinates to parametrize.

$$x = \rho\cos\theta\sin\phi,\ \ y = \rho\sin\theta\sin\phi,\ \ z = \rho\cos\phi$$

For this hemisphere $\rho = 2$ so

$$x = 2\cos\theta\sin\phi,\ \ y = 2\sin\theta\sin\phi,\ \ z = 2\cos\phi$$

where $0 \le \theta \le 2\pi,$ and $0 \le \phi \le \frac{\pi}{2}$.

4*b* [15 **pts**.] Give an iterated double integral equal to

$$\iint_S zdS$$

where *S* is the surface in part 3a. Do *not* evaluate your expression.

Solution:

$$\vec{r}(\theta,\phi) = 2\cos\theta\sin\phi\vec{i} + 2\sin\theta\sin\phi\vec{j} + 2\cos\phi\vec{k}$$

$$\iint_S zdS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}(2\cos\phi)\left|\vec{r}_\theta \times \vec{r}_\phi\right|d\phi d\theta$$

We need to calculate $\vec{r}_\theta \times \vec{r}_\phi$.

$$\vec{r}_\theta = -2\sin\theta\sin\phi\vec{i} + 2\cos\theta\sin\phi\vec{j}$$
$$\vec{r}_\phi = 2\cos\theta\cos\phi\vec{i} + 2\sin\theta\cos\phi\vec{j} - 2\sin\phi\vec{k}$$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi \end{matrix}$$

$$= -4\cos\theta\sin^2\phi\vec{i} + 0\vec{j} - 4\sin^2\theta\sin\phi\cos\phi\vec{k} - 4\cos^2\theta\cos\phi\sin\phi\vec{k} - 4\sin^2\phi\cos\theta\vec{j}$$

$$= -4\cos\theta\sin^2\phi\vec{i} - 4\sin^2\phi\cos\theta\vec{j} - 4\sin\phi\cos\phi\vec{k}$$

Thus

$$\left|\vec{r}_\theta \times \vec{r}_\phi\right| = \sqrt{16\cos^2\theta\sin^4\phi + 16\sin^4\phi\cos^2\theta + 16\sin^2\phi\cos^2\phi}$$

$$= \sqrt{16\sin^4\phi + 16\sin^2\phi\cos^2\phi} = 4\sqrt{\sin^2\phi\left(\sin^2\phi + \cos^2\phi\right)}$$

$$= 4\sqrt{\sin^2\phi} = 4\sin\phi$$

So

$$\iint_S z\,dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (2\cos\phi)\left|\vec{r}_\theta \times \vec{r}_\phi\right| d\phi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (2\cos\phi)4\sin\phi d\phi d\theta$$

$$= 8\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi d\phi d\theta$$

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$