Ma 227 **Exam III A Solutions** **12/4/12**

Name: ________________________________

Lecture Section: ________ **Recitation Section**: _______________

I pledge my honor that I have abided by the Stevens Honor System. ___________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

#4 _________

Total Score ________

1 [30 **pts**.] Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

where

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$$

and S is the surface $2x + 2y + z = 3$ in the first octant.
Solution: To parameterize S let $z = 3 - 2x - 2y$ then

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + (3 - 2x - 2y)\vec{k}$$

$$\vec{r}_x = 1\vec{i} + 0\vec{j} - 2\vec{k}$$

$$\vec{r}_y = 0\vec{i} + 1\vec{j} - 2\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} = 2\vec{i} + 2\vec{j} + 1\vec{k}$$

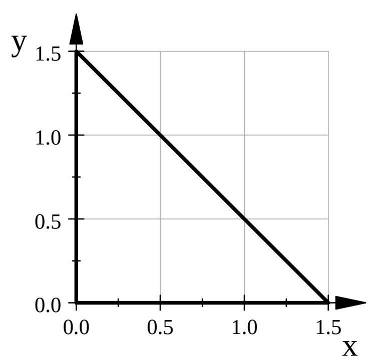
On S

$$\vec{F} = x\vec{i} + y\vec{j} + (3 - 2x - 2y)\vec{k}$$

so

$$\vec{F} \bullet \left(\vec{r}_x \times \vec{r}_y\right) = 2x + 2y + 3 - 2x - 2y = +3$$

The projection of S into the x,y-plane is the line $2x + 2y = 3$ or $y = \frac{3}{2} - x$. The region of integration is shown below



Thus

$$\iint_S \vec{F} \bullet \vec{n} dS = \int_0^{\frac{3}{2}} \int_0^{\frac{3}{2}-y} (+3) dx dy = 3\int_0^{\frac{3}{2}} \left(\frac{3}{2} - y\right) dy = 3\left(\frac{3}{2}y - \frac{y^2}{2}\right)_0^{\frac{3}{2}} = 3\left(\frac{9}{4} - \frac{9}{8}\right) = \frac{27}{8}$$

2 [30 **pts**.] Use Stokes' Theorem to evaluate

$$\int\int_S \text{curl}\,\vec{F} \cdot \vec{n} ds = \int\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} ds$$

for the vector $\vec{F} = y\vec{i} - x\vec{j}$, where S is the hemisphere $x^2 + y^2 + z^2 = 9, z \geq 0$. Use an outward normal.

Solution: Stokes Theorem says

$$\int\int_S \text{curl}\,\vec{F} \cdot \vec{n} ds = \int\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} ds = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

We calculate $\oint_{\partial S} \vec{F} \cdot d\vec{r}$. Now ∂S is the circle $x^2 + y^2 = 9,\ z = 0$. We parametrize this as

$$x = 3\cos t, \qquad y = 3\sin t, \qquad z = 0 \qquad 0 \leq t \leq 2\pi$$

On ∂S

$$\vec{F} = 3\sin t\,\vec{i} - 3\cos t\,\vec{j}$$

and

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k} = 3\cos t\,\vec{i} + 3\sin t\,\vec{j} + 0\vec{k}$$
$$\Rightarrow \vec{r}'(t) = -3\sin t\,\vec{i} + 3\cos t\,\vec{j}$$

Thus,

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-9\sin^2 t - 9\cos^2 t) dt = \int_0^{2\pi} (-9) dt = -18\pi$$

3 [20 **pts**.] Let S be the closed surface of the solid cylinder T bounded by the planes $z = 0$ and $z = 3$ and the cylinder $x^2 + y^2 = 4$.

Calculate the surface integral

$$\iint_S \vec{F} \cdot \vec{n} dS$$

where

$$\vec{F} = \left(x^2 + y^2 + z^2\right)\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

Solution: We use the Divergence Theorem which is

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T \nabla \cdot \vec{F} dV$$

$$\vec{F} = \left(x^2 + y^2 + z^2\right)\left(x\vec{i}\right) + \left(x^2 + y^2 + z^2\right)\left(y\vec{j}\right) + \left(x^2 + y^2 + z^2\right)\left(z\vec{k}\right)$$

$$\operatorname{div} \vec{F} = 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2 + x^2 + y^2 + 3z^2 = 5\left(x^2 + y^2 + z^2\right)$$

Then

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T 5\left(x^2 + y^2 + z^2\right) dV$$

Using cylindrical coordinates to evaluate the integral we have

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T 5\left(x^2 + y^2 + z^2\right) dV = \int_0^{2\pi} \int_0^2 \int_0^3 \left(5\left(r^2 + z^2\right) r\right) dz dr d\theta = 300\pi$$

4 [20 **pts**.] If

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Find A^{-1}.

Solution: We form $\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \to_{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \to_{-2R_1 + R_2 \text{ and } -R_1 + R_3} \begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3+(-1)R_2 \text{ and } R_1+4R_2} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{\left(-\frac{1}{2}\right)R_3} \begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \xrightarrow{-11R_3+R_1 \quad -3R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Check (not required) : $AA^{-1} = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Table of Integrals

$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$
$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$
$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$
$\int te^t dt = e^t(t-1) + C$
$\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$