**Ma 227** **Exam III A Solutions** **12/3/13**

**Name**: ______________________________

**1** [30 **pts**.] Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS$$

where

$$\vec{F} = x\vec{k}$$

and $S$ is the surface with parametrization

$$x = u^2 \quad y = v \quad z = u^3 - v^2 \quad 0 \le u \le 1 \quad 0 \le v \le 1$$

and oriented by an upward pointing normal.

Solution:

$$\vec{r}(u,v) = u^2\vec{i} + v\vec{j} + \left(u^3 - v^2\right)\vec{k}$$

so

$$\vec{r}_u = 2u\vec{i} + 3u^2\vec{k}$$
$$\vec{r}_v = \vec{j} - 2v\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 3u^2 \\ 0 & 1 & -2v \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 2u & 0 \\ 0 & 1 \end{matrix} = 2u\vec{k} - 3u^2\vec{i} + 4uv\vec{j} = -3u^2\vec{i} + 4uv\vec{j} + 2u\vec{k}$$

Setting $u = 1$ and $v = 0$ we have that $\vec{r}_u \times \vec{r}_v = -3\vec{i} + 2\vec{k}$ which is upward.

$\vec{F}(u,v) = u^2\vec{k}$ so

$$\vec{F}(u,v) \cdot (\vec{r}_u \times \vec{r}_v) = 2u^3$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \int_0^1 \int_0^1 2u^3 dv du$$

$$= \int_0^1 \left(2u^3\right) du = \frac{1}{2}$$

**2** [20 **pts**.] Let $S$ be any closed bounded surface with outward directed normal $\vec{n}$ in $x, y, z$ –space, and

$$\vec{F} = \left(z^2 + xy^2\right)\vec{i} + \cos(x+z)\vec{j} + \left(e^{-y} - zy^2\right)\vec{k}.$$

Evaluate

$$\iint_S \vec{F} \cdot \vec{n} dS$$

Solution: We use the Divergence Theorem

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T \nabla \cdot \vec{F} dV$$

to evaluate this surface integral.

$$\nabla \cdot \vec{F} = \operatorname{div} \vec{F} = y^2 - y^2 = 0$$

Thus

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_T \nabla \cdot \vec{F} dV = \iiint_T 0 dV = 0$$

where $T$ is the region enclosed by $S$.

**3** [20 **pts**.] Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

Solution:

$$\det(A - rI) = \begin{vmatrix} 2-r & -4 \\ -1 & -1-r \end{vmatrix} = -(2-r)(1+r) - 4$$
$$= (r-2)(r+1) - 4$$
$$= r^2 - r - 6 = (r-3)(r+2)$$

Thus the eigenvalues are $r = 3, -2$. The system

$$(A - rI)X = 0$$

is

$$(2-r)x_1 - 4x_2 = 0$$
$$-x_1 + (-1-r)x_2 = 0$$

For $r = 3$ we have

$$-x_1 - 4x_2 = 0$$
$$-x_1 - 4x_2 = 0$$

So $x_1 = -4x_2$. Letting $x_2 = 1$ we have the eigenvector $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $r = 3$. Letting $r = -2$ in the above system we have

$$4x_1 - 4x_2 = 0$$
$$-x_1 + x_2 = 0$$

so $x_1 = x_2$ and an eigenvector corresponding to $r = -2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

SNB check: $\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$, eigenvectors: $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow -2, \left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\} \leftrightarrow 3$

**4** [30 **pts**.] Consider the vector $\vec{F} = -y\vec{i} + 2x\vec{j} + (x+z)\vec{k}$ and let $S$ be the upper hemisphere

$$x^2 + y^2 + z^2 = 1, z \geq 0.$$

with an upward normal. Note:

$$\text{curl}\,\vec{F} = 0\vec{i} - \vec{j} + 3\vec{k}$$

Using spherical coordinates since $\rho = 1$, a parametrization of the hemisphere is

$$x = \sin\phi\cos\theta, \qquad y = \sin\phi\sin\theta, \qquad z = \cos\phi.$$

With this parametrization

$$\vec{r}(\phi,\theta) = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}$$

and a normal to the hemisphere is

$$\vec{r}_\phi \times \vec{r}_\theta = \sin^2\phi\cos\theta\vec{i} + \sin^2\phi\sin\theta\,\vec{j} + \sin\phi\cos\phi\vec{k}$$

Show this normal is upward, and use this information to verify Stokes' Theorem.
Solution: Stokes Theorem says

$$\int\int_S \text{curl}\,\vec{F}\cdot\vec{n}ds = \int\int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}ds = \oint_{\partial S}\vec{F}\cdot d\vec{r}$$

We calculate $\oint_{\partial S}\vec{F}\cdot d\vec{r}$ first. Now $\partial S$ is the circle $x^2 + y^2 = 1,\ z = 0$. We parametrize this as

$$x = \cos t, \qquad y = \sin t, \qquad z = 0 \qquad 0 \leq t \leq 2\pi$$

On $\partial S$

$$\vec{F} = -\sin t\,\vec{i} + 2\cos t\,\vec{j} + \cos t\vec{k}$$

and

$$\vec{r}(t) = x\vec{i} + y\vec{j} + z\vec{k} = \cos t\,\vec{i} + \sin t\,\vec{j} + 0\vec{k}$$
$$\Rightarrow \vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

Thus,

$$\oint_{\partial S}\vec{F}\cdot d\vec{r} = \int_0^{2\pi}(\sin^2 t + 2\cos^2 t)dt = \int_0^{2\pi}(1+\cos^2 t)dt = \left(t + \frac{1}{2}\cos t\sin t + \frac{1}{2}t\right)_0^{2\pi} = 3\pi$$

We now show that

$$\int\int_S \text{curl}\,\vec{F}\cdot\vec{n}ds = 3\pi$$

First of all
At $\phi = \frac{\pi}{2}$, $\quad \theta = 0$, $\vec{r}_\phi \times \vec{r}_\theta = \vec{i}$, which is uptward. Hence $\vec{r}_\phi \times \vec{r}_\theta$ is uptward.
Hence

$$\begin{aligned}
\operatorname{curl}\vec{F} \cdot \left(\vec{r}_\phi \times \vec{r}_\theta\right) &= \left(0\vec{i} - \vec{j} + 3\vec{k}\right) \cdot \left(\sin^2\phi\cos\theta\vec{i} + \sin^2\phi\sin\theta\vec{j} + \sin\phi\cos\phi\vec{k}\right) \\
&= -\sin^2\phi\sin\theta + 3\sin\phi\cos\phi
\end{aligned}$$

Thus we have

$$\begin{aligned}
\int\int_S \operatorname{curl}\vec{F} \cdot \vec{n}ds &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (-\sin^2\phi\sin\theta + 3\sin\phi\cos\phi)d\theta d\phi \\
&= \int_0^{\frac{\pi}{2}} \left(\sin^2\phi\cos\theta + 3\theta\sin\phi\cos\phi\right)_{\theta=0}^{\theta=2\pi} d\phi \\
&= 0 + 6\pi\int_0^{\frac{\pi}{2}} \sin\phi\cos\phi d\phi = 6\pi\frac{\sin^2\phi}{2}\Bigg|_0^{\frac{\pi}{2}} = 3\pi \text{ as before.}
\end{aligned}$$

# Table of Integrals

| |
|---|
| $\int \sin^2 x dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$ |
| $\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$ |
| $\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$ |
| $\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$ |
| $\int te^t dt = e^t(t-1) + C$ |
| $\int t^2 e^t dt = e^t\left(t^2 - 2t + 2\right) + C$ |