Ma 227 **Exam III B Solutions** **4/25/05**

Name: ______________________________

Lecture Section: ______________________________ **Lecturer**: ________________________

I pledge my honor that I have abided by the Stevens Honor System. ________________________________

You may not use a calculator, cell phone, or computer while taking this exam. All work must be shown to obtain full credit. Credit will not be given for work not reasonably supported. When you finish, be sure to sign the pledge.

Score on Problem #1 ________

#2 ________

#3 ________

Total Score ________

1a [15 pts.] Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, if

$$\vec{F}(x,y) = yz\vec{i} - xz\vec{k}$$

where C is the plane path $x(t) = 2t - 1, y(t) = 2 - 4t, z(t) = t,\ 0 \le t \le 1.$
Solution:

$$\vec{r}(t) = (2t-1)\vec{i} + (2-4t)\vec{j} + t\vec{k} \quad 0 \le t \le 1$$

so

$$\vec{r}'(t) = 2\vec{i} - 4\vec{j} + \vec{k}$$

$$\vec{F}(t) = (2-4t)t\vec{i} - (2t-1)t\vec{k}$$

Thus

$$\vec{F} \cdot \vec{r}'(t) = 4t - 8t^2 - 2t^2 + t = -10t^2 + 5t$$

Hence

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(5t - 10t^2\right)dt = \left[\frac{5}{2}t^2 - \frac{10}{3}t^3\right]_0^1 = \frac{5}{2} - \frac{10}{3} = -\frac{5}{6}$$

1b [15 pts.] Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Show that

$$\nabla(\ln r) = \frac{\vec{r}}{r^2}$$

Solution: $r = \sqrt{x^2 + y^2 + z^2}$

$$\begin{aligned}
\nabla \ln r &= \frac{\partial \ln r}{\partial x}\vec{i} + \frac{\partial \ln r}{\partial y}\vec{j} + \frac{\partial \ln r}{\partial z}\vec{k} \\
&= \frac{1}{2}\left(\frac{2x}{r\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{2y}{r\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{2z}{r\sqrt{x^2+y^2+z^2}}\vec{k}\right) \\
&= \frac{1}{r^2}\vec{r}
\end{aligned}$$

2a $\left[20\ \textbf{pts.}\right]$ Find a function $\Phi(x,y,z)$ such that $\nabla\Phi = \vec{F}$, where

$$\vec{F}(x,y,z) = \left(3x^2 + z\right)\vec{i} + \left(3y^2 - z\right)\vec{j} + \left(3z^2 - y + x\right)\vec{k}$$

Solution:

$$\Phi_x = 3x^2 + z$$

$\Rightarrow$

$$\Phi = x^3 + xz + g(y,z)$$

$\Rightarrow$

$$\Phi_y = \frac{\partial g}{\partial y} = 3y^2 - z$$

$\Rightarrow$

$$g(y,z) = y^3 - yz + h(z)$$

Thus

$$\Phi = x^3 + xz + y^3 - yz + h(z)$$

and

$$\Phi_z = x - y + h'(z) = 3z^2 - y + x$$

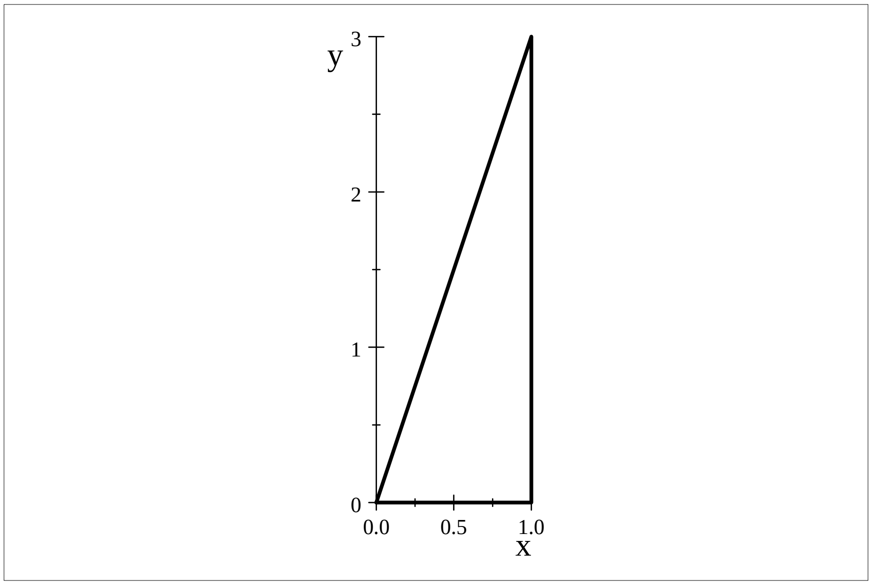
so $h'(z) = 3z^2 \Rightarrow h(z) = z^3 + K$ and

$$\Phi = x^3 + xz + y^3 - yz + z^3 + K$$

2b [**20 pts**.] Verify that Green's Theorem is true for the line integral

$$\oint_C x^2 y dx + xy^2 dy$$

where C is the triangle with vertices $(0,0),(1,0),(1,3)$. Sketch the triangle.
Solution: The curve C and the region of integration are shown below
$(0,0,1,0,1,3,0,0)$



The line joining $(0,0)$ to $(1,2)$ is $y = 3x$. Now we must show

$$\oint_C P dx + Q dy = \iint_R (Q_x - P_y) dA$$

$P = x^2 y$ and $Q = xy^2$. Let C_1 be the segment joining $(0,0)$ and $(1,0)$, C_2 the segment joining $(1,0)$ and $(1,3)$ and C_3 the segment joining $(1,3)$ and $(0,0)$.

$$\oint_C x^2 y dx + xy^2 dy = \int_{C_1} x^2 y dx + xy^2 dy + \int_{C_2} x^2 y dx + xy^2 dy + \int_{C_3} x^2 y dx + xy^2 dy$$

$$= \int_0^1 0 dx + \int_0^3 (1) y^2 dy + \int_1^0 \left[x^2 (3x) dx + x(3x)^2 (3dx) \right]$$

$$= \left[\frac{y^3}{3} \right]_0^3 + \left[\frac{3}{4} x^4 + 27 \frac{x^4}{4} \right]_1^0 = 9 - \frac{3}{4} - \frac{27}{4} = \frac{6}{4} = \frac{3}{2}$$

Also

$$\iint_R (Q_x - P_y) dA = \iint_R \left(y^2 - x^2 \right) dA = \int_0^1 \int_0^{3x} \left(y^2 - x^2 \right) dy dx$$

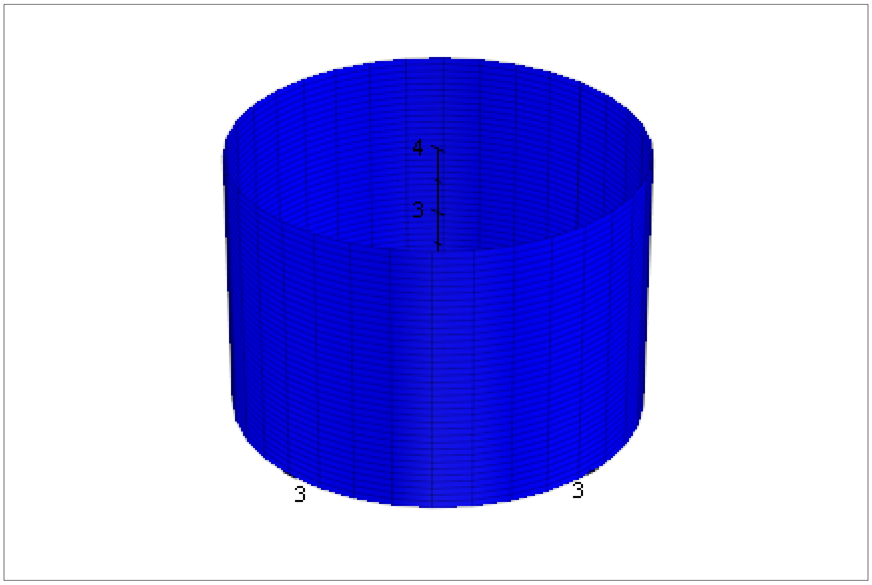
$$= \int_0^1 \left[\frac{y^3}{3} - x^2 y \right]_0^{3x} dx = \int_0^1 \left[9x^3 - 3x^3 \right] dx = \left[\frac{9x^4}{4} - \frac{3x^4}{4} \right]_0^1 = \frac{6}{4}$$

3 a [**10 pts**.] Let S be the portion of the cylinder $x^2 + y^2 = 9$ that lies between $z = 0$ and $z = 4$. Use cylindrical coordinates to

to give a parametrization of S.
Solution: Let $x = 3\cos\theta, y = 3\sin\theta, z = z,\ \ 0 \le z \le 4,\ \ 0 \le \theta \le 2\pi$ or

$$\vec{r}(\theta, z) = 3\cos\theta\vec{i} + 3\sin\theta\vec{j} + z\vec{k} \quad 0 \le z \le 4,\ 0 \le \theta \le 2\pi$$

$(3, \theta, z)$



3 b [**20 pts**.] Give an expression for

$$\iint_S y dS$$

where S is the surface in part 3a. Do *not* evaluate your expression.
Solution: For a surface given by

$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

that

$$\int\int_S f(x,y,z)ds = \int\int_G f(u,v)|\vec{r}_u \times \vec{r}_v| dudv,$$

where G is the image of the surface S in the u,v -plane. Letting $u = \theta$ and $v = z,$ we have

$$\vec{r}(\theta, z) = 3\cos\theta\vec{i} + 3\sin\theta\vec{j} + z\vec{k}$$

and

$$\vec{r}_\theta(\theta, z) = -3\sin\theta\vec{i} + 3\cos\theta\vec{j}$$
$$\vec{r}_z(\theta, z) = \vec{k}$$

Thus

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\cos\theta\vec{i} + 3\sin\theta\vec{j}$$

so

$$|\vec{r}_\theta \times \vec{r}_z| = 3$$

Thus

$$\iint_S y dS = \int_0^{2\pi}\int_0^4 3\sin\theta(3)dzd\theta = 9\int_0^{2\pi}\int_0^4 \sin\theta dzd\theta$$